SOURCES OF COMPLEX DYNAMICS IN TWO-SECTOR GROWTH MODELS*

Michele BOLDRIN
University of California, Los Angeles, CA 90024, USA

Raymond J. DENECKERE
Northwestern University, Evanston, IL 60208, USA

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This paper develops a tractable multisectoral dynamic equilibrium model and provides a fairly complete analysis of the dynamics that may arise along the intertemporal competitive equilibrium path. Despite the fact that the environment displays neither random nor deterministic variability, the model may produce oscillations in aggregate variables such as output, consumption, and investment. We show that these oscillations can be exactly periodic, of any finite period \( n \), and even totally aperiodic (chaotic). We characterize the parameter values for which each of these cases occur, but also provide some strong global asymptotic stability results.

1. Introduction

Economists used to interpret the business cycle as a nonequilibrium phenomenon. In fact, they regarded it as the prime example of inconsistencies between predictions of Walrasian models and the reality of capitalist market economies [see, e.g., Goodwin (1982), Kaldor (1940), and Kalecki (1971)]. More recently, various students of the business cycle have shown that simple, aggregate, competitive models may be able to explain some relevant features of observed time series, provided one introduces stochastic forces that displace the economy from its stable stationary position. Lucas (1987) contains an excellent overview of this line of literature.

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Here, we adopt the modern framework of intertemporal competitive equilibrium, but at the same time dispense with the need for stochastic forces in order to produce and sustain oscillations in aggregate variables. We obtain endogenous oscillations in spite of the very restrictive deterministic framework we adopt. However, our results should not be taken to mean that we advocate a purely deterministic explanation of the cycle. Indeed, there may be good economic (as well as technical) reasons to allow for random disturbances in dynamic economic models. The significance of our work is rather that it underlines the role of nonlinearities in explaining the persistence of fluctuations in aggregate economic data.

Our model economy consists of identical consumers that maximize the discounted value of consumption over an infinite time horizon. Production is carried out by two neoclassical industries, one producing consumption goods and the other producing capital goods. Markets for factors and commodities clear at all points in time, and equilibrium prices are perfectly known at the beginning of time. With these assumptions, our model has a unique competitive equilibrium. The competitive equilibrium path is summarized by a dynamical system \( \tau_{t+1} = \tau(k_t) \) that describes the evolution of the aggregate capital stock over time. We show that for certain parameter values the dynamical system \( \tau \) exhibits chaotic orbits.

That oscillating paths may appear in standard models of competitive equilibrium over time has already been demonstrated by Benhabib and Nishimura (1985) for the case of period-two cycles, and by Deneckere and Pelikan (1986), Boldrin and Montrucchio (1986), and Boldrin (1989) for the chaotic case. What distinguishes the current paper from the research in Deneckere and Pelikan (1986) and Boldrin and Montrucchio (1986) is that we do not construct 'artificial' economies that exhibit a pre-chosen dynamics in equilibrium. Rather, we start with a specification of technology and preferences and derive the implied dynamics. While this was also the object of Boldrin's (1989) study, the focus there is more on providing criteria for the existence of chaotic paths in abstract two-sector models (see also Deneckere (1988)). Here we specify two analytically tractable functional forms for the production technologies and carry out a complete parametric analysis of the resulting dynamics.

As the analysis below will reveal, both a high degree of impatience (i.e., small discount factors) and a huge difference in the productivity of the two factors (labor and capital) are required in order to produce chaotic dynamics. More precisely, we find chaotic solutions when capital is very productive in both sectors, but labor is not. This productivity difference accounts for the...
huge variations of the capital stock over the course of a ‘typical’ cycle. It does not, however, explain why the economy finds it profitable to oscillate, rather than to proceed along a smooth averaged path, exploiting the concavity in production. In order to understand the economic forces that drive oscillations, we need to analyze the relative profitability of the two sectors at each point in time. This profitability, in turn, depends on the relative capital–labor intensities of the sectors. Since capital–labor intensity reversals play a critical role in our analysis, we will describe the process in some detail.

In our model, the technology of the investment sector is of the constant coefficients type; the other is of the Cobb–Douglas variety. The efficient capital–labor ratio is thus fixed at some level $0 < \gamma < 1$ in the investment sector, but is completely free to move in the consumption sector. We assume that labor is supplied exogenously and normalized to one at all time. The aggregate capital–labor ratio is then equal to the aggregate capital stock. Static efficiency and full employment of inputs require that during periods in which the aggregate capital stock is less than $\gamma$, the consumption industry must have a capital–labor ratio less than the aggregate one, and hence less than $\gamma$. Thus, when the aggregate capital stock is between zero and $\gamma$, the investment sector is more capital-intensive than the consumption good sector. Similar reasoning shows that the reverse must be true when $k$, exceeds $\gamma$.

Let us now compare two different periods during which the aggregate stocks $k$ and $k'$ are less than $\gamma$, with $k' > k$. A straight application of the Rybczynski theorem shows that we should observe a higher investment output in the period associated with $k'$ than in the period associated with $k$. If $k$ and $k'$ are the stocks of the two adjacent periods, this process will yield a path of increasing capital stocks. This explains the rising portion of the cycle.

After a finite number of periods, the aggregate capital stock will exceed $\gamma$. At that point the same efficiency and market-clearing conditions will result in the consumption sector being more capital-intensive. The substitution effect along the aggregate production possibility frontier then makes it profitable to increase the consumption output and decrease the investment output. This ‘recession’ phase will continue until the aggregate stock falls below $\gamma$. At that stage, the cycle is complete.

Oscillations of the type described above may be exactly periodic, of some finite period $n$, or totally aperiodic (chaotic), depending on the parameter values. We show that any of these cases is, in fact, possible. Because of the strict concavity of the production possibility frontier this wandering of $k$, will imply huge variations in relative prices and rates of returns. A high level of discounting is then needed in order to eliminate the arbitrage possibilities that would otherwise emerge. This, in turn, suggests that one may avoid ‘unrealis-

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*Ivar Ekeland and Jose Scheinkman were first to conjecture that the parametric example we study here could produce chaotic optimal paths; see Scheinkman (1984)."
tic' levels of discounting if appropriate portions of increasing returns are introduced in the aggregate technology. This point is elaborated upon in Deneckere and Pelikan (1984) and is also discussed further in the conclusion.

The reader should not be tempted to believe that factor-intensity reversals are typical only of the specific technologies we have chosen. On the contrary, the same phenomenon will be present with any pair of distinct constant elasticity of substitution production functions, as long as they are not both Cobb–Douglas, linear, or input–output. Our choice has been motivated solely by concerns of tractability. One last observation: we have stressed capital-intensity reversals as the driving force behind cyclical movements. We made this choice because it makes the underlying economic forces very transparent. But, as the discussion in section 2 suggests, this is not strictly necessary. The case in which the consumption sector is always more capital-intensive and the capital stock does not depreciate instantaneously can also produce chaotic dynamics [see also section 5 and Boldrin (1989)].

The paper proceeds as follows: in section 2 we set up the general model and recall some results spread about in the literature. Section 3 introduces the specific parameterization and studies the dynamical system \( k_{t+1} = \tau_\delta(k_t) \). In section 4, we introduce a related map \( h_\delta \), which allows us to prove some global asymptotic stability results. We also derive analytical expressions for \( \tau_\delta \) in certain regions of the parameter space. In section 5, we analyze a version of the model without instantaneous depreciation. Section 6 concludes the paper.

2. A competitive two-sector economy

Below, we study the dynamic behavior of a simple aggregate model of competitive equilibrium over time. We depart from the traditional Cass–Koopmans one-sector model of growth only in assuming a nonlinear transformation frontier between consumption and investment. This is equivalent to the Jones–Uzawa framework, where consumption and capital are different commodities produced in different sectors. The formal model will be briefly outlined in this section; a more detailed account may be found in Boldrin (1989).

Let \( k_t \) denote the stock of capital at date \( t \) \((t = 0, 1, 2, \ldots)\); \( k_t \) also represents the aggregate capital–labor ratio. The aggregate amount of labor supplied is thus exogenous and normalized to one for all \( t \). Capital evolves according to the equation \( k_{t+1} = \mu k_t + y_t \), where \( y_t \) is the output from the investment sector and \((1 - \mu)\) is the capital depreciation rate \((0 \leq 1 - \mu \leq 1)\). Consumption is produced according to the process \( c_t = F^1(l^1_t, k^1_t) \). Similarly, for investment, \( y_t = F^2(l^2_t, k^2_t) \). Finally, there is a single (representative) agent who owns the entire capital stock, supplies labor and capital to both industries via competitive markets, and maximizes the sum of discounted consumption.
over his (infinite) lifetime. We make the following assumption on the production functions:

**H.1:** For each \( i \) \( (i = 1, 2) \), \( F^i: \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \) is continuous, concave, homogeneous of degree one, and increasing in both arguments.

Standard results on the equivalence of competitive equilibria and Pareto optima establish that the output of the consumption sector may be expressed as a function of the existing capital stock and the output of the investment good: \( c_i = T(k_i, y_i) \). The production possibility frontier \( T \) is the solution to

\[
\begin{align*}
\max & \quad F^1(l^1, k^1), \\
\text{subject to} & \quad F^2(l^2, k^2) \geq y, \\
& \quad l^1 + l^2 \leq 1, \\
& \quad k^1 + k^2 \leq k, \\
& \quad l^1, k^1, l^2, k^2 \geq 0.
\end{align*}
\]

Assumption H.1 implies that \( T \) is concave, increasing in \( k \), and decreasing in \( y \). It will be of class \( C^2 \) on its domain of definition when each \( F^i \) is \( C^2 \). We also assume that as a consequence of decreasing returns:

**H.2:** There exists a \( \bar{k} > 0 \) such that \( F^2(1, k) < (1 - \mu)k \) for all \( k > \bar{k} \), and \( F^2(1, k) > (1 - \mu)k \) for all \( k < \bar{k} \). Also, \( F^2(l^2, 0) = 0 \) for all \( l^2 \in \mathbb{R}_+ \).

Let \( y = f(k) \) denote the solution to \( T(k, y) = 0 \). Then H.2 implies that for \( k > \bar{k} \), \( f(k) < (1 - \mu)k \), and for \( k < \bar{k} \), \( f(k) > (1 - \mu)k \). Thus, no capital stock exceeding \( \bar{k} \) can be sustained, and, without loss of generality, we may restrict our analysis to the feasible set \( D \):

\[
D = \{(k_i, k_{i+1}) \in \mathbb{R}^2_+ \quad \text{s.t.} \quad 0 \leq k_I \leq \bar{k} \}
\]

and \( \mu k_i \leq k_{i+1} \leq f(k_i) + \mu k_i \).

\( D \) is convex and has nonempty interior as long as \( f \) is not identically zero on \([0, \bar{k}]\). The competitive equilibrium \( \{c_i, y_i, q_i, r_i, w_i, l_i^1, l_i^2, k_i^1, k_i^2\}_{i=0}^{\infty} \), where \( q_i \) is the price of capital, \( w_i \) the wage rate, and \( r_i \) the rental rate (with consumption taken as a numeraire), may then be derived from the sequence of optimal capital stocks \( \{k_i\}_{i=0}^{\infty} \) that solve

\[
W_\delta(k_0) = \max \sum_{i=0}^{\infty} V(k_i, k_{i+1}) \delta',
\]

(P)
subject to

\[ (k_t, k_{t+1}) \in D, \]
\[ V(k_t, k_{t+1}) = T(k_t, k_{t+1} - \mu k_t), \]
\[ k_0 \text{ given in } [0, \tilde{k}], \]

using the following relations (which hold either by definition or as a condition for equilibrium):

\[ c_t = V(k_t, k_{t+1}), \] (1)
\[ y_t = k_{t+1} - \mu k_t, \] (2)
\[ q_t = \delta W'_\theta(k_{t+1}) = -V'_2(k_t, k_{t+1}), \] (3)
\[ r_t = V'_1(k_t, k_{t+1}), \] (4)
\[ w_t = V(k_t, k_{t+1}) + q_t(k_{t+1} - \mu k_t) - r_t k_t, \] (5)
\[ l'_i = l'_i(k_t, k_{t+1} - \mu k_t), \quad i = 1, 2, \] (6)
\[ k'_i = k'_i(k_t, k_{t+1} - \mu k_t), \quad i = 1, 2. \] (7)

The functions \( k'(\cdot) \) and \( l'(\cdot) \) above are the solutions to (T), and \( W_\theta: [0, \tilde{k}] \to \mathbb{R} \) is the value function associated with (P). With \( W'_\theta(\cdot) \) denoting the derivative of this function, in (3) we are implicitly assuming that \( T \) is at least \( C^1 \) and applying the result of Benveniste and Scheinkman (1979). In fact we will make the stronger hypothesis:

H.3: \( T: D \to \mathbb{R}_+ \) is of class \( C^2 \) on \( \text{int}(D) \).

Let \( \tau_\theta(\cdot): [0, \tilde{k}] \to [0, \tilde{k}] \) be the policy function associated with \( W_\theta(\cdot) \), i.e.,

\[ \tau_\theta(x) = \arg \max \{ V(x, y) + \delta W(y); (x, y) \in D \}. \] (8)

The optimal sequence \( \{ k_i \} \) is generated by the discrete dynamical system \( k_{i+1} = \tau_\theta(k_i) \), starting at \( k_0 \). While analytic formulae for \( \tau_\theta(\cdot) \) and \( W_\theta(\cdot) \) are generally not available, a qualitative analysis can nevertheless be performed. To this end, we first collect some theoretical results.

Fact 1. Let \( (x, y) \in \text{Int}(D) \) be a point on the policy function, i.e., \( y = \tau_\theta(x) \). Then, if \( V_{12}(x, y) > 0 \) (\(< 0\)), the policy function is locally increasing (decreasing) at \( (x, y) \). Furthermore, if \( (y, \tau_\theta(y)) \in \text{Int}(D) \), then \( V_{12}(x, y) > 0 \) (\(< 0\),
Proof. See Benhabib and Nishimura (1985).

Fact 2 (turnpike). For every strictly concave $V$ (and associated feasible set $D$), there exists a $\tilde{\delta} < 1$ such that for all $\delta \in [\tilde{\delta}, 1)$ the dynamical system $k_{t+1} = \tau_\delta(k_t)$ is globally asymptotically stable, i.e., there exists a unique $k^*$ such that $k_t(k_0) \to k^*$ as $t \to \infty$, for every $k_0 \neq 0$.

Proof. See McKenzie (1986), Scheinkman (1976), and Deneckere and Pelikan (1986).

Fact 3. Assume $V_{12}(x, y) < 0$ for $(x, y) \in \tilde{D} \subset \text{Int}(D)$. Let $(x^*(\delta), x^*(\delta)) \in \tilde{D}$ satisfy $\tau_\delta(x^*(\delta)) = x^*(\delta)$ for $\delta \in [\delta^-, \delta^+] \subset (0, 1)$. Assume there exists $\delta^0 \in (\delta^-, \delta^+)$ such that, evaluated at $(x^*(\delta), x^*(\delta))$,

(i) $V_{22} + \delta V_{11} - (1 + \delta)V_{12} = 0$ for $\delta = \delta^0$,

(ii) $V_{22} + \delta V_{11} - (1 + \delta)V_{12} > 0$ for $\delta \in [\delta^-, \delta^0)$,

(iii) $V_{22} + \delta V_{11} - (1 + \delta)V_{12} < 0$ for $\delta \in (\delta^0, \delta^+]$.

Then there exists a period-two orbit for $\tau_\delta$ for all $\delta$ in some (right or left) neighborhood of $\delta^0$.

Proof. See Benhabib and Nishimura (1985).

Fact 4. Under hypotheses H.1 and H.3, we have:

(i) $T_{12}(x, y) > 0$ for all $(x, y) \in D$ such that $(k^1/l^1)(x, y) < (k^2/l^2)(x, y)$, and

(ii) $T_{12}(x, y) < 0$ for all $(x, y) \in D$ such that $(k^1/l^1)(x, y) > (k^2/l^2)(x, y)$.

Furthermore, if $(\tilde{x}, \tilde{y}) \in D$ is such that $T_{12}(\tilde{x}, \tilde{y}) = 0$, then $T_{12}(\tilde{x}, y) = 0$ for all $y$ feasible from $\tilde{x}$.

Proof. See Benhabib and Nishimura (1985) and Boldrin (1989).

Fact 5. Suppose that $V_{12}(x, y) \neq 0$ for all $(x, y) \in \text{Int}(D)$ and suppose that $\tau_\delta$ is interior. Then every trajectory converges either to a steady state or to a cycle of period two.

As pointed out in the Introduction, the objective of this paper is to study conditions under which our simple two-sector competitive economy displays persistent and endogenous fluctuations, of both the periodic and the aperiodic type. A word of caution regarding the use of the term 'chaos' is appropriate here. We will say that $\tau_8$ displays 'topological chaos' if it conforms to the definitions of Li and Yorke (1975). This notion of chaos is often contrasted with 'ergodic chaos', i.e., the existence of a unique ergodic invariant measure for $\tau_8$ that is absolutely continuous with respect to the Lebesgue measure (with or without a positive Lyapounov exponent). For further details on these delicate matters the reader should consult Collet and Eckmann (1980), Devaney (1986), or Guckenheimer and Holmes (1983).

The last formal result we need gives a set of computable sufficient conditions for the existence of topological chaos in a two-sector economy.

**Fact 6.** Assume there exists a $k^* \in (0, \bar{k})$ such that $V_{12}(k^*, \cdot) = 0$. Then $\tau_8$ has topological chaos for all $\delta \in (0,1)$ that satisfy the three conditions:

(i) $V_2(x, k^*) + \delta V_1(k^*, \cdot) = 0$ has two roots, $k_1 \in (0, k^*)$ and $k_2 \in (k^*, \bar{k})$,

(ii) $V_2(x, k_1) + \delta V_1(k_1, k^*) = 0$ has a root $k_3 \in [k^*, \bar{k}]$, and

(iii) $V_2(x, k_3) + \delta V_1(k_3, k_1) = 0$ has at least one real root.

**Proof.** See Boldrin (1989).

Fact 2 is the classical turnpike theorem: we will not make much use of it, but to note that there exists an upper bound on the set of $\delta$ that may produce oscillating behavior. Fact 3 shows that when $\tau_8$ is downward-sloping around an optimal steady state (OSS), then a cycle of period two may bifurcate from the OSS when it loses stability. Since the information necessary to verify the hypothesis of this result is local in nature, we can use it to detect orbits of period two. This will be the first (or last) step of a bifurcation cascade leading to chaos in our model. A simple generalization of Fact 3 will also allow us to detect the existence of orbits of period 4 and, potentially, of any orbit with period $2^n$. Fact 5 shows that a nonmonotonic $\tau_8$ (and, if $\tau_8$ is interior, a change in the sign of $V_{12}$) is necessary, although not sufficient, for complicated dynamics. We will adopt the terminology of Deneckere and Pelikan (1986) here and say that a dynamical system displays 'simple dynamics' if its attractors are either fixed points or points of period two. Facts 1 and 4 link the slope of $\tau_8$ to factor-intensity conditions. For the case in which $\mu = 0$ (i.e., the capital stock lasts only one period) the causal relation is clear: $\tau_8$ is increasing when the investment sector has a higher capital–labor ratio than the consumption sector, decreasing in the opposite case and flat at the reversal points.
When $\mu \neq 0$ we can easily see that
\[ V_{12}(k_t, k_{t+1}) = T_{12}(k_t, k_{t+1} - \mu k_t) - \mu T_{22}(k_t, k_{t+1} - \mu k_t). \] (9)

The slope of $\tau_8$ depends, therefore, also on $\mu$ and the sensitivity of the price of capital to variations in the output of the investment sector. The critical point of $\tau_8$, when it exists, will not necessarily coincide with a factor-intensity reversal, and will not be independent of $\delta$, as in the case when $\mu = 0$. On the other hand, note that $\tau_8$ may now be nonmonotonic even in the absence of a capital-intensity reversal: this is true if $T_{12}$ is negative everywhere and if both $\mu$ and $T_{22}$ are 'large enough'.

Finally, Fact 6 tells us how to check for chaos when the critical point $k^*$ is independent of $\delta$ [and thus of $\tau_8(k^*)$]. Obviously, this is the case when $\mu = 0$. In the parameterized model we study below, the result is also applicable to the case $\mu > 0$, but that is a consequence of the functional forms chosen.

3. The model

In this section (and in section 4), we analyze the case $\mu = 0$. Section 5 reports on the case $\mu > 0$.

3.1. Technology and preferences

Assume that $F^1$ is of the Cobb--Douglas variety and that $F^2$ is Leontief:
\[ c_t = F^1(l_t^1, k_t^1) = (l_t^1)^\alpha (k_t^1)^{1-\alpha}, \quad \alpha \in (0, 1), \] (10)
\[ y_t = F^2(l_t^2, k_t^2) = \min \{ l_t^2, k_t^2 / \gamma \}, \quad \gamma \in (0, 1). \] (11)

Since $k_{t+1} = y_t$ and $l_t^1 + l_t^2 = 1$ for all $t$, the feasible set and the PPF, respectively, reduce to
\[ D = \left\{ \left( x, y \right) \in [0, 1] \times [0, 1] \ \text{s.t.} \ 0 \leq y \leq \min \{ 1, x / \gamma \} \right\}, \] (12)
\[ T(k_t, k_{t+1}) = (1 - k_{t+1})^\alpha (k_t - \gamma k_{t+1})^{1-\alpha}. \] (13)

Straightforward economic intuition suggests that $k^* = \gamma \in (0, 1)$ is the unique value of $k$ where a capital--labor-intensity reversal takes place: as the (efficient) $k^2/l^2$ ratio is fixed at $\gamma$, an economy-wide capital--labor ratio less than $\gamma$ will make the consumption sector less capital-intensive and the opposite will be true for $k_t$ larger than $\gamma$. We take utility to be linear in consumption. Since labor is supplied inelastically, $V(k_t, k_{t+1}) = T(k_t, k_{t+1})$. Facts 1 and 4 and (13) then imply that $\tau_8$ is increasing on $[0, \gamma)$ and decreasing on $(\gamma, 1]$. Also,
from (11), note that \( \tau_\delta(0) = 0 \) for all \( \delta \), and that any choice \( k_{i+1} \in [0, 1] \) is feasible when \( k_i \geq \gamma \).

3.2. The dynamic equilibria

As shown in section 2, the optimal sequence \( \{k_i\} \) must solve

\[
\max \sum_{t=0}^{\infty} \delta^t (1 - k_{t+1})^\alpha (k_t - \gamma k_{t+1})^{1-\alpha},
\]

subject to

\[
0 \leq k_{t+1} \leq \min\{1, k_t/\gamma\},
\]

\( k_0 \) given in \([0, 1]\).

By computing the partial derivatives of \( V \), we see that \((x, \tau_\delta(x)) \in \text{Int} \ D\) for all \( x \in (0, 1)\), except possibly at \( x = \gamma \). Note also that \( V \) loses strict concavity in \( x \) along the upper boundary of \( D \), and strict concavity in \( y \) along the vertical line \( x = \gamma \). Finally, \( V_1 V_{22} - V_{12}^2 = 0 \) for all \((x, y) \in D\).

Clearly, there is no hope of finding analytical expressions for either \( W_s \) or \( \tau_\delta \). Nevertheless, we can use the Euler equation to extract some information on the local behavior of \( \tau_\delta \) for those parameter values where it is interior to \( D \). The Euler equation is

\[
\delta (1 - \alpha) \left( \frac{1 - k_{t+1}} {k_t - \gamma k_{t+1}} \right)^\alpha = \left( \frac{1 - k_t} {k_{t-1} - \gamma k_t} \right)^\alpha \left[ \gamma (1 - \alpha) + \alpha \left( \frac{k_{t-1} - \gamma k_t} {1 - k_t} \right) \right].
\]

From (15) we may, first of all, conclude that there exists at most one OSS different from zero, namely,

\[
k^* = \frac{(\delta - \gamma)(1 - \alpha)}{(\delta - \gamma)(1 - \alpha) + \alpha(1 - \gamma)}.
\]

The position of \( k^* \) relative to \( \gamma \) is of some importance, as \( \tau_\delta \) is increasing on \([0, \gamma]\) and decreasing on \((\gamma, 1]\). Observe that \( k^* \leq \gamma \) as \( \delta \leq \gamma/(1 - \alpha) \). The next proposition uses this information to state a global asymptotic stability result for \( \delta \leq \gamma/(1 - \alpha) \).

**Proposition 1.** For \( 0 < \delta \leq \gamma \), the optimal path \( k_t \) converges to zero for any initial condition \( k_0 \) in \([0, 1]\) and no interior OSS exists. For \( \gamma < \delta \leq \gamma/(1 - \alpha) \),
there exists a unique interior OSS $k^*$ defined by (16) and the optimal path converges to $k^*$, for any $k_0$ in $(0, 1]$.

**Proof.** The first part is obvious. Since 0 is the only fixed point for $\tau_\delta$, we have $k_{i+1} = \tau_\delta(k_i) < k_i$, and thus $k_i \to 0$ as $i \to \infty$. If, on the other hand, $\delta \in (\gamma, \gamma/(1 - \alpha)]$, then $k^*$ lies in $(0, \gamma]$ and the origin is unstable. Thus, for $x \in (0, k^*)$, we have $\tau_\delta(x) > x$, and the trajectory from $x$ converges to $k^*$. Conversely, if $x \in (k^*, 1]$, $\tau_\delta(x) < x$, and thus the entire interval $[k^*, 1]$ is attracted to $k^*$.

Observe that Proposition 1 implies global asymptotic stability if $\alpha \geq 1 - \gamma$, independently of $\delta$! Let us now turn to the case where $\delta > \gamma/(1 - \alpha)$. It is well known from the literature on optimal growth theory [see, for example, McKenzie (1986) and Scheinkman (1976)] that when $k^*$ is locally stable for $\tau_\delta$, i.e., $|d\tau_\delta(k)/dk|_{k=k^*} < 1$, the second-order system produced by the Euler equation (15) has a local saddle-point structure at $k^*$. This means that of the two eigenvalues of the characteristic polynomial:

$$
\delta V_{12}(k^*, k^*) \lambda^2 + \left[ V_{22}(k^*, k^*) + \delta V_{11}(k^*, k^*) \right] \lambda + V_{12}(k^*, k^*) = 0,
$$

(17)

associated with the linearization of (15), one lies inside and one lies outside the unit circle. In fact, the smaller eigenvalue corresponds to $\tau_\delta'(k^*)$ [see Deneckere and Pelikan (1984)]. For our example, (17) reduces to

$$
\delta \left[ (\gamma - k^*)/(1 - k^*) \right] \lambda^2 - \left[ \delta + (\gamma - k^*)^2/(1 - k^*)^2 \right] \lambda + (\gamma - k^*)/(1 - k^*) = 0,
$$

(18)

from which we may compute

$$
\lambda_1 = \alpha/\left[ \gamma - (1 - \alpha) \delta \right] < 0, \quad \lambda_2 = \left[ \gamma - (1 - \alpha) \delta \right] / \alpha \delta < 0.
$$

(19)

The signs of the expressions in (19) follow from (18) and the fact that $k^* > \gamma$ whenever $\delta > \gamma/(1 - \alpha)$. We also see that

$$
\lambda_1 \in (-\infty, -1) \quad \text{for} \quad \delta \in (\gamma/(1 - \alpha), (\alpha + \gamma)/(1 - \alpha)),
$$

$$
\lambda_1 \in (-1, 0) \quad \text{for} \quad \delta > (\alpha + \gamma)/(1 - \alpha),
$$

(20)
and
\[
\lambda_2 \in (-1, 0) \quad \text{for} \quad \delta \in \left(\frac{\gamma}{1-\alpha}, 1\right), \quad \text{if} \quad \alpha \geq \frac{1}{2},
\]
\[
\lambda_2 \in (-1, 0) \quad \text{for} \quad \delta \in \left(\frac{\gamma}{1-\alpha}, \frac{\gamma}{1-2\alpha}\right), \quad \text{(21)}
\]
\[
\lambda_2 \in (-\infty, -1) \quad \text{for} \quad \delta > \frac{\gamma}{1-2\alpha}, \quad \text{if} \quad \alpha < \frac{1}{2}.
\]

Hence, we have proven:

**Proposition 2.** When \( \alpha \geq (1-\gamma)/2 \), the OSS \( k^* \) is locally asymptotically stable for all \( \delta \in \left(\frac{\gamma}{1-\alpha}, 1\right) \). When \( \alpha < (1-\gamma)/2 \), it is stable for \( \delta \in \left(\frac{\gamma}{1-\alpha}, \frac{\gamma}{1-2\alpha}\right) \cup \left(\frac{\alpha + \gamma}{1-\alpha}, 1\right) \).

One should note that Proposition 2 states a local result only. Intuition suggests that \( k^* \) may in fact be globally asymptotically stable, but a proof of this requires additional analysis (see section 4).³

A natural question now arises: What happens when \( \delta \in \left(\frac{\gamma}{1-2\alpha}, \frac{\alpha + \gamma}{1-\alpha}\right) \)? What dynamics does the competitive equilibrium sequence \( k, \) then display? A partial answer is the following proposition:

**Proposition 3.** Let \( \alpha < (1-\gamma)/2 \). Then the policy function \( \tau_\delta \) has a cycle of period two for \( \delta \) in a neighborhood of \( \delta^- = \frac{\gamma}{1-2\alpha} \) and \( \delta^+ = \frac{\alpha + \gamma}{1-\alpha} \). These cycles are locally stable when they exist for \( \delta \in (\delta^-, \delta^+) \), and unstable in the other cases.

**Proof.** To get existence one needs only to apply Fact 3 to our model. The sign of \( V_{22} + \delta V_{11} - (1+\delta)V_{12} \), evaluated at \( k^* \), is opposite to that of \( [((k^* - \gamma)/(1-k^*))^2 - (1+\delta)((k^* - \gamma)/(1-k^*)) + \delta \). It is easy to see that this expression is zero at \( \delta = \delta^- \) or \( \delta^+ \), negative for \( \delta \in (\delta^-, \delta^+) \), and positive otherwise. Therefore, cycles of period two exist both around \( \delta^- \) and \( \delta^+ \). Stability follows from standard results in dynamical systems theory. A simple proof can be found in Benhabib and Nishimura (1985, corollary 1).

³Our discussion above also implies the following stability intervals with respect to the other parameters of the model: \( k^* \in (\gamma, 1) \) is stable for \( \alpha \in (0, (\delta - \gamma)/(1+\delta)) \cup ((\delta - \gamma)/2\delta, (\delta - \gamma)/\delta) \), and for \( \gamma \in (0, (1-\alpha)\delta - \alpha) \cup (\delta(1-2\alpha), \delta(1-\alpha)) \).
contention is that, for suitable $\alpha$ and $\gamma$, there exists an interval $(\delta^*, \delta^{**}) \subset (\delta^-, \delta^+)$ at which $\tau_6$ has (at least) topological chaos. We demonstrate this at the end of this section. We also believe that the emergence of chaos follows the classical ‘period-doubling bifurcation pattern’ as $\delta \to \delta^*$ from the left or $\delta \to \delta^{**}$ from the right. Without complete knowledge of $\tau_6$ this claim cannot be proven. However, evidence from the simulations we have run supports this contention. Here we only show that a second flip bifurcation may lead to an orbit of period four. We use the same logic behind Fact 3 and Proposition 3.

**Proposition 4.** Let $x(\delta), y(\delta)$ denote an interior period-two orbit of $\tau_6$ for $\delta$ values in $(\delta^-, \delta^+)$. Let $V_{ij} = V_{ij}(x(\delta), y(\delta))$ and $V_{ij} = V_{ij}(y(\delta), x(\delta))$, $i, j = 1, 2$. Assume there exists an interval $[\delta^-, \delta^+] \subset (\delta^-, \delta^*)$ and a $\delta_0 \in (\delta^-, \delta^*)$ such that the function $G(\delta) = V_{22}^* V_{22} + \delta^2 V_{11}^* V_{11} + \delta (V_{11}^* V_{22} - V_{12}^*) + \delta (V_{11} V_{22} - V_{12}) + (1 + \delta^2) V_{12}^* V_{12}$ satisfies:

\[
\begin{cases}
  > 0 & \text{for } \delta \in [\delta^-, \delta^0), \\
  = 0 & \text{for } \delta = \delta^0, \\
  < 0 & \text{for } \delta \in (\delta^0, \delta^+). 
\end{cases}
\]

Then there exists a period-four orbit for $\tau_6$ bifurcating from $(x(\delta), y(\delta))$ at $\delta = \delta^0$.

**Proof.** See the appendix.

Proposition 4 implies the following corollary, whose proof also appears in the appendix:

**Corollary.** Let $(x(\delta), y(\delta))$ be a period-two point for our model, and suppose that it exists for all $\delta$ in $(\delta^-, \delta^*)$ and that $x(\delta) < \gamma$ and $y(\delta) > \gamma$. Then a cycle of period four exists for all values $\delta_0 \in (\delta^-, \delta^*)$ at which either one of the following two equations holds:

\[
\begin{align*}
  (i) & \quad (x(\delta^0) - \gamma)/(1 - x(\delta^0)) = -\delta^2 (1 - y(\delta^0))/y(\delta^0)\gamma, \\
  (ii) & \quad (x(\delta^0) - \gamma)/(1 - x(\delta^0)) = -(1 - y(\delta^0))/y(\delta^0)\gamma.
\end{align*}
\]

To verify the presence of such bifurcations in our model, consider the example $\alpha = 0.03, \gamma = 0.09$. Proposition 2 implies that the steady state $k^*$ is locally stable when $\delta$ lies in the interval $[0.0928, 0.0957]$. For discount factors in $[0.0957, 0.0974]$ stable period-two orbits are present, verifying Proposition 3. At $\delta = 0.0974$, the period-two orbit $x^* = 0.0738, y^* = 0.3980$ bifurcates into a stable period-four orbit, which exists for $\delta \in [0.0974, 0.0978]$. In fact, our
simulation results reveal that successive bifurcations eventually lead to chaos when $\delta$ reaches the value 0.099. This chaos exists for $\delta \in [0.099, 0.112]$, as can be checked directly by applying Fact 6 of section 2 to our model. Fig. 1 describes the evolution of the policy function $\tau_\delta$ for $\alpha = 0.03$ and $\gamma = 0.09$, and fig. 2 depicts the set of $(\alpha, \gamma)$ parameters for which topological chaos is present for some $\delta \in (0, 1)$. As expected, the concavity of $V$ implies that only extreme values of the parameters can yield chaotic dynamics.

One might suspect that part of the reason that such extreme values of the parameters are needed stems from the fact that the elasticity of substitution
between capital and labor in the consumption-good sector is fairly large. To investigate this issue, we also ran simulations for a tractable generalization of our model, suggested by Boldrin (1989). This generalization retains the Leontief technology for the investment sector, but allows for a CES in the consumption sector. Thus, \( F^1(l^1, k^1) = [\alpha(l^1)^\rho + (1 - \alpha)(k^1)^\rho]^{1/\rho} \), which approaches (10) as \( \rho \to 0 \). The elasticity of substitution, \( \sigma \), for the CES is equal to \( 1/(1 - \rho) \); negative values of \( \rho \) thus permit much smaller values of \( \sigma \). The simulations revealed that chaotic optimal paths do arise for this model as well, and that when \( \sigma \) is fairly low, chaos appears for values of the discount factor roughly three times larger than those found for the Cobb-Douglas model. A typical example has \( \alpha = 0.2, \gamma = 0.2, \delta = 0.25, \) and \( \rho = -0.5 \). Since the values for the discount factor at which chaos appears in the Cobb-Douglas model are themselves approximately 100 times larger than the ones found in the artificial economies constructed by Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986), no definite conclusion can be drawn, at this stage, as to whether a model of this type could produce chaotic dynamics at more reasonable values of the discount factor.

4. The first-order difference equation \( h_\delta \)

4.1. The relationship between \( \tau_\delta \) and \( h_\delta \)

For \( t = 0, 1, 2, \ldots \) consider the ratio \( z_t = (1 - k_{t+1})/(k_t - \gamma k_{t+1}) \). From eqs. (6), (7), (10), and (11), we see that \( z_t = l_t^1/k_t^1 \), i.e., \( z_t \) is the time \( t \) labor-capital ratio in the consumption sector. The advantage of working with the variable \( z_t \) is that this reduces (15) from a second-order difference equation in \( k_t \) to a first-order difference equation in \( z_t \):

\[
z_t = z_{t-1} \left[ \frac{\gamma}{\delta} + \frac{\alpha}{\delta(1 - \alpha)} z_{t-1}^{\alpha} \right]^{1/\alpha}.
\]

More compactly, we may write \( z_t = h_\delta(z_{t-1}) \).

The purpose of this section is to investigate whether any useful information on the dynamics of the unknown policy function \( \tau_\delta \) can be derived from the study of \( h_\delta \). Before elaborating on the relationship between \( \tau_\delta \) and \( h_\delta \), we must first establish some properties of the map \( h_\delta \).

To simplify notation, let \( a = \gamma/\delta \) and \( b = \alpha/((1 - \alpha)\delta) \), so that \( h(z) = z[(a + bz^{-1})^{1/\alpha}] \). Since we have already characterized \( \tau_\delta \) for \( \delta \) values less than \( \gamma \), we will confine attention here to the case \( a < 1 \). It is apparent that \( h \) is decreasing on \((0.1/\gamma)\) and increasing on \((1/\gamma, \infty)\), with \( \lim_{z \to \infty} h(z) = \infty \). In fact, if we let \( \theta = (\gamma/\delta)^{1/\alpha} < 1 \), we see that \( \lim_{z \to \infty} [h(z) - \theta z] = 0 \), so that \( h \) asymptotes to \( \theta z \) as \( z \) approaches infinity. Furthermore, \( h \) has a unique fixed
point \( z^* = h(z^*) \), which satisfies \( z^* = a[(\delta - \gamma)(1 - \gamma)]^{-1} \). A typical graph of \( h \) [with \( \delta \geq \gamma/(1 - \alpha) \)] is depicted in fig. 3. One may also check that the map \( h_\delta : \mathbb{R}_+ \to \mathbb{R}_+ \) is of at least class \( C^3 \) on any compact subset of \( \mathbb{R}_+ \) and that it has negative Schwarzian derivative [for a definition, see Devaney (1986, p. 68)] when \( 0 < \alpha < 1/2 \).

For \( \delta \leq \gamma/(1 - \alpha) \), we have \( z^* \geq 1/\gamma \), and thus \( h^n(z) \to z^* \) for all \( z \in (0, \infty) \). This confirms Proposition 1 of section 3. Henceforth, we will confine attention to values of \( \delta \) exceeding \( \gamma/(1 - \alpha) \), so that \( z^* < 1/\gamma \) and \( h(1/\gamma) < 1/\gamma \).

Consider now the interval \( I = [z(1/\gamma), z(2/\gamma)] \). It is easy to see that \( I \) is an invariant set for \( h \). Moreover, the basin of attraction of \( I \) under \( h \) is \((0, \infty)\), i.e., for every \( z_0 \in (0, \infty) \) there exists \( N \) sufficiently large such that \( h^n(z_0) \in I \), for all \( n \geq N \). Thus, we may restrict our study of \( h \) to the interval \( I \), i.e., consider the dynamical system \( h_\delta : I \to I \), for \( \delta \in (\gamma/(1 - \alpha), 1) \).

Let us now turn to the relationship between \( \tau_0 \) and \( h_\delta \). For any initial point \( k_0 \in [0, 1] \), pick an arbitrary \( k_1 \) that is feasible from \( k_0 \). The pair \( (k_0, k_1) \) corresponds to a choice of \( z_0 \in (0, \infty) \). With these initial conditions, we may run (15) forward to produce a candidate optimal path \( \{k_t\} \) or, equivalently, run (22) forward to produce a candidate optimal path \( \{z_t\} \). Because \( z_t \) will be in \( I \) for sufficiently large \( t \), the associated sequence \( \{k_t\} \) will be uniformly bounded and therefore satisfy the transversality condition \( \lim_{t \to +\infty} \delta_t q_t k_t = 0 \). Hence we cannot rule out nonoptimal \( \{k_t\} \) (or \( \{z_t\}) by appealing to the transversality condition. Since interior optimal paths must satisfy the Euler equation (15) [or (22)], and since optimal paths are unique, we conclude that all but one choice of \( k_1 \) from \( k_0 \) will induce a sequence \( \{z_t\} \) that corresponds to a sequence \( \{k_t\} \) which is infeasible, i.e., that does not satisfy \( 0 \leq k_{t+1} \leq \min\{1, k_t/\gamma\} \). We will not give a complete characterization of the relationship...
between the policy function and the map \( h_\delta \) here. Rather, we will content ourselves to observing that when the graph of \( \tau_\delta \) is interior every periodic point of \( \tau_\delta \) induces a periodic point for \( h_\delta \) of equal (or lower) periodicity. Because period-two points play a central role in describing the dynamics of one-dimensional systems, we will study those in detail below.

**Proposition 5.** Let \((w, z)\) be a period-two orbit for \( h \), i.e., \( h(w) = z \) and \( h(z) = w \). Then the pair \((x, y)\) defined by

\[
x = \frac{1 - \gamma z}{(1 - \gamma z)(1 - \gamma w) - wz}, \quad y = \frac{1 - \gamma w}{(1 - \gamma z)(1 - \gamma w) - wz},
\]

is an optimal cycle for \( \tau_\delta \) if and only if the pair \((w, z)\) satisfies one of the following restrictions:

(i) \( 0 < w < 1/(1 + \gamma) \), \( 0 < z < 1/(1 + \gamma) \).

(ii) \( 1/(1 + \gamma) < w \leq \min\{1/\gamma, h^2(1/\gamma)\} \), \( 1/(1 + \gamma) < z \leq h^2(1/\gamma) \).

**Proof.** If \((x, y)\) is a period-two point for \( \tau_\delta \), the labor–capital ratios \( w \) and \( z \) must satisfy \( w = (1 - y)/(x - yx) \) and \( z = (1 - x)/(y - yx) \). Inverting these relationships yields the stated expressions for \( x \) and \( y \). Some simple calculations now show that feasibility, i.e., \( 0 \leq x \leq \min(y/\gamma, 1) \) and \( 0 \leq y \leq \min(x/\gamma, 1) \) implies either \( 0 < w, z < 1/(1 + \gamma) \) or \( z > 1/(1 + \gamma) \) and \( 1/(1 + \gamma) < w < 1/\gamma \). Finally, the restrictions \( w < \min(1/\gamma, h^2(1/\gamma)) \) and \( z \leq h^2(1/\gamma) \) in (ii) follow from the fact that \((w, z)\) and \((z, w)\) must lie in \( I \times I \).

Graphically, the two situations are illustrated in fig. 4.

**Corollary.** Case (i) of Proposition 5 is possible only if \( \delta > (\alpha + \gamma)/(1 - \alpha) \), and case (ii) only if \( \delta \in (\gamma/(1 - \alpha), (\alpha + \gamma)/(1 - \alpha)) \).

**Proof.** For case (i) to apply, we need \( z^* = \alpha/((\delta - \gamma)(1 - \alpha) < 1/(1 + \gamma) \), and vice versa for case (ii). This inequality is easily rewritten as \( \delta > (\alpha + \gamma)/(1 - \alpha) \). We showed that \( \delta \leq \gamma/(1 - \alpha) \) implies global stability of \( z^* \). This yields the lower endpoint of the interval in case (ii).

4.2. Simple dynamics for the map \( h_\delta \)

In section 3, we proved that the steady state \( k^* \) is globally asymptotically stable when \( \delta \leq \gamma/(1 - \alpha) \). We also saw that \( k^* \) was locally asymptotically stable when \( \delta \in [\gamma/(1 - \alpha), 1) \) and \( \alpha \geq (1 - \gamma)/2 \) and when \( \delta \in [\gamma/(1 - \alpha), \gamma/(1 - 2\alpha)) \cup (\alpha + \gamma)/(1 - \alpha), 1) \) and \( \alpha < (1 - \gamma)/2 \). We will now strengthen some of these results.
Proposition 6. Suppose $\delta \geq \gamma/(1 - \alpha)$ and $(1 - \alpha) + \alpha[\delta(1 - \alpha)/\gamma]^{1/\alpha} < \delta (1 - \alpha)/\gamma^2$. Then $h_\delta$ has simple dynamics.

Proof. Under the conditions of the proposition, $0 < h(1/\gamma) < h^2(1/\gamma) < 1/\gamma$. Thus, the map $h_\delta$ confined to the trapping region $I \times I$ is strictly monotone (decreasing) and hence can only display simple dynamics.

Corollary 1. Under the conditions of Proposition 6, the policy function $\tau_\delta(\cdot)$ has simple dynamics.

Proof. Assume first that graph $\tau_\delta \subset \text{Int } D$, i.e., that $\tau_\delta(\gamma) < 1$. In that case, the Euler equation (15) must hold along all optimal trajectories. Observe now that the trapping region for $\tau_\delta$ corresponds to the trapping region for $h_\delta$. Observe also that the trapping region for $h_\delta$, $I \times I$, is entirely located on its downward-sloping branch. Now any value of $k$ such that $z(k) = (1 - \tau_\delta(k))/(k - y\tau_\delta(k))$ has $(z(k), h_\delta(z(k))) \in I \times I$ must satisfy $k > \gamma$. The trapping region for $\tau_\delta$ is thus also located on its downward-sloping branch.

Next, suppose that $\tau_\delta(\gamma) = 1$; the Euler equation then holds from every value of the initial capital stock, except possibly at $k = \gamma$. Some reflection now shows that the fact that the Euler condition may hold as an inequality at $k = \gamma$ implies that the dynamics of $z$, is now given by a ‘chopped off’ version of $h_\delta$, i.e., that $z_{t+1} = \max\{\omega, h_\delta(z_t)\}$ for some choice of $\omega$. The trapping region for this modified function will be a strict (nonempty) subset of the trapping region for $h_\delta$, and hence the same reasoning as above can be applied.
Corollary 2. The conditions of Proposition 6 are satisfied for all \( \delta \in (\gamma/(1 - \alpha), 1) \) when \( \alpha \geq (1 - \gamma)/2 \), and for \( \delta \in (\gamma(1 - \alpha), \gamma/(1 - 2\alpha)) \) when \( \alpha < (1 - \gamma)/2 \).

Proof. Let \( v = \delta(1 - \alpha)/\gamma \). By assumption, \( v > 1 \). We may rewrite the condition of Proposition 6 as \( \phi(v) = (1 - \alpha) + \alpha v^{1/\alpha} - v^2 < 0 \). Observe that \( \phi(1) = 0 \), and that, when \( \alpha \geq 1/2 \), \( \phi(v) = v^{(1/\alpha)-1} - 2v \leq v - 2v < 0 \). When \( \alpha < 1/2 \), there exists a unique \( \delta(\alpha) \) such that \( \phi(v) \geq 0 \) as \( v \geq \delta(\alpha) \). Observe that \( \delta \leq \gamma(1 - 2\alpha) \) implies \( v \leq (1 - \alpha)/(1 - 2\alpha) \) independently of the relationship between \( \alpha \) and \( \gamma \), and that \( \alpha \geq (1 - \gamma)/2 \) implies \( v \leq (1 - \delta(\alpha)) \leq (1 - \alpha)/(1 - 2\alpha) \). One can now show that \( \phi((1 - \alpha)/(1 - 2\alpha)) < 0 \), proving the desired result for \( \alpha < 1/2 \). \( \blacksquare \)

One puzzling feature of the map \( h_\delta(\cdot) \) should be noticed: we may easily calculate the slope of \( h_\delta \) at \( z^* \):

\[
h_\delta'(z^*) = \left[ \gamma - \delta(1 - \alpha) \right]/\alpha \delta. \tag{23}
\]

The slope of \( h_\delta \) thus coincides with \( \lambda_2 \), one of the eigenvalues of the Euler equation linearized at the steady state \( k^* \). The second eigenvalue \( \lambda_1 \) apparently gets lost in the transformation \((k_t, k_{t+1}) \rightarrow z_t \). This implies that the (locally) stabilizing effect of \( \lambda_1 \) on the dynamics of \( k_t \) when \( \delta > (\alpha + \gamma)/(1 - \alpha) \) and \( \alpha < (1 - \gamma)/2 \) is not transmitted to the orbits of \( z_t \). In particular, for such parameter values, \( \tau_\delta(\cdot) \) is locally stable around \( k^* \), whereas \( h_\delta(\cdot) \) is locally unstable around \( z^* \). We solve this apparent puzzle in Proposition 7.

Proposition 7. For \( \delta \in ((\alpha + \gamma)/(1 - \alpha), 1) \) the policy function satisfies

\[
\tau_\delta(k_t) = (1 - \lambda_1) k^* + \lambda_1 k_t \quad \text{for all} \quad k_t \in [\gamma, 1]. \tag{24}
\]

Furthermore, the steady state \( k^* \) is globally asymptotically stable.

Proof. From (20) we know that \( \lambda_1 \) is stable in the assumed range for \( \delta \). Global stability then follows immediately from the functional form (24).

Using (16) and (19), we may rewrite (24) as

\[
k_{t+1} = \frac{(\delta - \gamma)(1 - \alpha)}{\delta(1 - \alpha) - \gamma} - \frac{\alpha}{\delta(1 - \alpha) - \gamma} k_t.
\]

To show that this is \( \tau_\delta(k_t) \), we only need to check that the pairs \((k_t, k_{t+1})\) so defined satisfy the Euler equation for all \( t \). But observe that

\[
\frac{1 - k_{t+1}}{k_t - \gamma k_{t+1}} = z_t = \frac{\alpha(k_t - \gamma)}{(\delta - \gamma)(1 - \alpha)(k_t - \gamma)} = z^*.
\]
Since any sequence $\tau^n$ stays in $[\gamma, 1]$ for all $k_0 \in [\gamma, 1]$ this proves the desired result. ■

Proposition 7 allows one to derive an explicit expression for $\tau_\delta$, not just on $[\gamma, 1]$ but actually on all of $[0, 1]$. Indeed, let $y \in [\gamma, 1]$. Since $\tau_\delta$ is monotone on $[\gamma, 1]$, with $\tau_\delta(\gamma) = 1$, there exist pre-images $x$ of $y$ in $[0, y]$. Let $p$ be the largest pre-image of $y$, i.e., $p = \sup\{x: \tau_\delta(x) = y\}$. Then for $x \in (p, y)$ the Euler equation holds:

$$V_2(x, y) + \delta(1 - \alpha)(z^*)^\alpha = 0. \tag{25}$$

In (25) we made use of (15) and the fact that $(1 - \tau_\delta(y))/(y - \gamma \tau_\delta(y)) = z^*$ (see Proposition 7). We conclude that the policy function is linear on $[p, y)$, with a slope $k$ that can be determined from (25). Now let $q = \inf\{x: \tau_\delta(x) = y\}$. The monotonicity of $\tau_\delta(\cdot)$ implies that $[q, p]$ is an interval. For $x \in (\tau^{-1}(p), q)$, the Euler equation holds again, and so

$$V_2(x, y) + \delta(1 - \alpha)z^\alpha = 0, \tag{26}$$

where $z = (1 - \tau_\delta(y))/(y - \gamma \tau_\delta(y))$ is constant since $\tau_\delta(\cdot)$ is linear on $(p, y)$. Thus, $\tau_\delta(\cdot)$ is linear on $[\tau_\delta^{-1}(p), q]$, with a slope that can be calculated from (26). In fact, it is easily seen that for all $x < q$, both $(x, \tau_\delta(x))$ and $(\tau_\delta(x), \tau_\delta^2(x))$ are in Int $D$, so that the Euler equation holds everywhere on $(0, q)$. Repeatedly taking pre-images using the Euler equation (26) then yields the entire functional form for $\tau_\delta(\cdot)$.

Observe that when $\delta = (\alpha + \gamma)/(1 - \alpha)$, Proposition 7 states that $\tau_\delta(\cdot)$ has a continuum of period-two cycles! Let $\rho = (\gamma/(1 + \gamma))^{\alpha}$. For $\delta \in [\rho(\alpha + \gamma)/(1 - \alpha), (\alpha + \gamma)/(1 - \alpha)]$ simulations indicate that a globally stable period-two orbit exists which lives on the boundary of the set $D$, namely $\tau_\delta(1) = \gamma$. We may also prove that $h_\delta$ has period-two cycles for $\delta$-values in a neighborhood of $\gamma/(1 - 2\alpha)$:

**Proposition 8.** $h_\delta$ has a stable period-two cycle when $\delta \in (\gamma/(1 - 2\alpha), \gamma/(1 - 2\alpha) + \varepsilon)$ for some $\varepsilon > 0$.

**Proof.** At $\delta = \gamma/(1 - 2\alpha)$, we have $h_\delta'(z^*) = -1$, $h_\delta''(z^*) \neq 0$, and $d/d\delta[h_\delta^2(z^*)]_{\delta = \gamma/(1 - 2\alpha)} \neq 0$. We may thus apply the flip bifurcation theorem [see, e.g., Devaney (1986, p. 89)] to obtain existence of a cycle of period two in an open neighborhood of $\gamma/(1 - 2\alpha)$. To see that the cycle occurs for $\delta$-values greater than $\gamma/(1 - 2\alpha)$, we observe that $h_\delta(z^*)$ and $(\partial/\partial \delta)h_\delta(z^*)$ have

---

4Observe that for $y \in [q, p]$ the Euler equation need not hold, and hence that $\tau_\delta(\cdot)$ is only weakly monotone (see Fact 1).
opposite signs at $\delta = \gamma/(1 - 2\alpha)$. Stability of the cycle follows from the fact that $z^*$ changes stability at $\delta = \gamma/(1 - 2\alpha)$. ■

Obviously, the cycle we recover here for $h_\delta$ corresponds to the cycle we obtain for $\tau_\delta$ in section 3, Proposition 3. The bifurcation theorem referred to in the proof above implies that the two-cycle will be close to $z^*$ when $\delta$ is close to $\gamma/(1 - 2\alpha)$. Simulations reveal that as $\delta$ increases, the cycles move away from $z^*$, and that eventually one of its points lies on the upward-sloping branch of $h_\delta$. A necessary condition for this to occur is that the trapping region $I \times I$ contains part of the upward-sloping branch of $h_\delta$, i.e., $h_\delta^3(1/\gamma) > 1/\gamma$. Letting $v = \delta(1 - \alpha)/\gamma$ and using Proposition 6, we see that this is equivalent to $(1 - \alpha) + \alpha v^{1/\alpha} > v^2$.

5. The model without instantaneous depreciation

When $\mu$ is greater than zero, the one-period return function becomes

$$V(x, y) = (1 + \mu x - y)^\alpha (b x - \gamma y)^{1-\alpha}, \quad (27)$$

where $b = 1 + \gamma \mu$. The feasible set is now

$$D = \{ (x, y) \in [0, 1/(1 - \mu)] \times [0, 1/(1 - \mu)] \}$$

s.t. $\mu x \leq y \leq \mu x + \min\{1, x/\gamma\}$. \quad (28)

Again, we may show that $V$ is of class $C^2$ on $\bar{D} = \text{Int}(D) \setminus \{x = \gamma\}$, and strictly concave in each variable, everywhere on the interior of $D$, except for the points $x = \gamma$, where $V_{22} = 0$, and $y = b$, where $V_{11} = 0$. As before, we also have $V_{11}V_{22} - V_{12}^2 = 0$ for all $(x, y) \in D$. The unique interior steady state can be easily computed from the Euler equation:

$$k^* = \frac{(\delta b - \gamma)(1 - \alpha)}{(\delta b - \gamma)(1 - \alpha)(1 - \mu) + \alpha(1 - \delta \mu)(b - \gamma)}. \quad (29)$$

Observe that (29) reduces to (16) when $\mu = 0$ (and $b = 1$). Observe also that $\delta b > \gamma$, i.e., $\delta > \gamma/(1 + \gamma \mu)$ is now required to guarantee that $k^*$ be bounded away from zero. Hence, an interior steady state will be present for $\delta$ values smaller than in the $\mu = 0$ case.

There are now two sets of points at which $V_{12}(x, y) = 0$. The first is, as before, $x = \gamma$ and the second is $y = b = 1 + \gamma \mu$. Since $1 + \gamma \mu < k = (1 - \mu)^{-1}$,
we need to check what happens at \( y = 1 + \gamma \mu \). It is not difficult to see that as long as \( \tau_8 \) is interior the value \( y = 1 + \gamma \mu \) will never be crossed, so that \((x, \tau_8(x))\) will always be confined to the region \([0, 1 + \gamma \mu] \times [0, 1 + \gamma \mu]\). Thus, the critical point is again at \( x = \gamma \), independently of \( \delta \). This is clearly coincidental: in general, when \( \mu \) is positive, the critical point of \( \tau_8 \) will depend on \( \delta \) [see Boldrin (1989)]. We may also compute that \( k^* \geq \gamma \) as \( \delta \geq \gamma/(1 + \gamma \mu - \alpha) \). Again, the critical value of \( \delta \) at which \( k^* \) moves on the downward-sloping branch of \( \tau_8 \) is smaller than when \( \mu = 0 \). The comparative statics of \( k^* \) with respect to \( \alpha, \gamma \), and \( \delta \) is the same as before and, consistent with economic intuition, \( \partial k^*/\partial \mu > 0 \). The eigenvalues of the linearized Euler equation at \( k^* \) are now

\[
\lambda_1 = \frac{b - k^*}{\gamma - k^*} = \mu + \frac{\alpha(1 - \delta \mu)(b - \gamma)}{\alpha \gamma (1 - \delta \mu)(b - \gamma) - (\delta b - \gamma)(1 - \alpha)[1 - \gamma(1 - \mu)]}, \tag{30a}
\]

\[
\lambda_2 = \frac{\gamma - k^*}{b - k^*} \delta^{-1} = \frac{\alpha \gamma (1 - \delta \mu)(b - \gamma) - (\delta b - \gamma)(1 - \alpha)[1 - \gamma(1 - \mu)]}{\alpha(1 + \gamma \mu)(1 - \delta \mu)(b - \gamma) - (\delta b - \gamma)(1 - \alpha)[1 - \gamma(1 - \mu)]} \delta^{-1}. \tag{30b}
\]

Some straightforward algebra shows that both eigenvalues are negative for \( \delta > \gamma/(1 - \alpha + \gamma \mu) \) and that their modulus behaves as in the case \( \mu = 0 \). In other words, there exists a pair \( \gamma/(1 - \alpha + \gamma \mu) < \delta^- < \delta^+ < 1 \) such that

\[
\lambda_1 \in (-\infty, -1) \quad \text{for} \quad \delta \in (\gamma/(1 - \alpha + \mu \gamma), \delta^+), \tag{31a}
\]

\[
\lambda_1 \in (-1, 0) \quad \text{for} \quad \delta > \delta^+, \tag{31b}
\]

and

\[
\lambda_2 \in (-1, 0) \quad \text{for} \quad \delta \in (\gamma/(1 - \alpha + \gamma \mu), \delta^-), \tag{32a}
\]

\[
\lambda_2 \in (-\infty, -1) \quad \text{for} \quad \delta > \delta^- \tag{32b}
\]

The expressions for \( \delta^- \) and \( \delta^+ \) in terms of \( \alpha, \gamma, \mu \) are, unfortunately, rather
Important differences with the case \( p = 0 \) nevertheless exist. In fact, simulations reveal that the chaotic solutions disappear rapidly as \( p \) increases from the full depreciation value towards one. For example, with \( (Y = 0.3, y = 0.2, \delta = 0.025) \) chaos exists for \( p \in [0, 0.09] \), but vanishes when \( p \geq 0.1 \). It is easy to understand why chaos disappears in this example when capital depreciates less quickly. The high level of capital productivity means that the rising portion of the cycle is confined to a small region of very low levels of the capital stock; unless depreciation is rapid, it will never be optimal to let

\[ M. \text{Boldrin and R.J. Deneckere, Sources of complex dynamics} \]

long and not very informative; therefore, we chose not to replicate them here. The analogy between these formulae and the ones previously obtained for the simple model \( \mu = 0 \) should convince the reader that all of the analysis in section 3 can be reproduced for the case \( \mu > 0 \) as well. Period-doubling bifurcations will be present for appropriate values of \( \alpha, \gamma, \delta, \) and \( \mu \). Also, because \( V_{12}(\gamma, y) = 0 \) for all \( y \in [0, \bar{k}] \), Fact 6 can be applied again (with the provision that \( k_i > \mu k_{i-1} \) for \( i = 2, 3 \)).

Important differences with the case \( \mu = 0 \) nevertheless exist. In fact, simulations reveal that the chaotic solutions disappear rapidly as \( \mu \) increases from the full depreciation value towards one. For example, with \( \alpha = 0.3, \gamma = 0.2, \) and \( \delta = 0.025 \) chaos exists for \( \mu \in [0, 0.09] \), but vanishes when \( \mu \geq 0.1 \). It is easy to understand why chaos disappears in this example when capital depreciates less quickly. The high level of capital productivity means that the rising portion of the cycle is confined to a small region of very low levels of the capital stock; unless depreciation is rapid, it will never be optimal to let

![Fig. 5. Evolution of the policy function \( \tau_k \) as a function of the depreciation rate \( (1 - \mu) \) for \( \alpha = 0.03, \gamma = 0.02, \) and \( \delta = 0.025 \).](image)
the capital stock fall to such a low level. Interestingly enough, chaos reappears as \( \mu \) moves towards one. In fact, when \( \mu \) is larger than 0.9 (i.e., depreciation ratios are on the order of ten percent or less), we find period-three orbits for \( \alpha = 0.3, \gamma = 0.02, \) and \( \delta = 0.025. \) The policy function for this case is illustrated in fig. 5. Observe that the mechanism generating chaos is now completely different from the one operative at \( \mu = 0. \) Indeed, the trapping region now lies completely to the right of the point of capital-intensity reversal \( \gamma. \) This type of chaos, therefore, does not rely on the presence of capital-intensity reversals. Rather, it exploits the interaction between the downward-sloping portion of the policy function and the depreciation constraint \( k_t = \mu k_{t-1}. \)

6. Conclusion

In this paper, we analyzed a simple general equilibrium model which produced unique, but sometimes cyclical and even chaotic paths for aggregate variables such as output, consumption, and investment. Despite the fact that an analytical expression for the policy function was often unavailable, we were able to characterize the dynamic behavior of our economy in terms of its basic parameters: \( \alpha \) (the labor share of income in the consumption sector), \( \gamma \) (the capital–labor ratio in the investment sector), and \( \delta \) (the discount factor). For many values of the parameters, the unique steady state was shown to be globally asymptotically stable. For other values of the parameters, we obtained a unique period-two point, which was globally attractive. Successive bifurcations then led to a chaotic regime, but only for rather unrealistic values of the parameters. Since our model is testable, we ran some simulations for the chaotic regimes and compared their statistical properties to those of the postwar U.S. economy [for details, the reader is urged to consult the working paper precursor of this article, Boldrin and Deneckere (1987)]. While at a qualitative level the simulated time series behaved fairly well, quantitatively the results were much worse than those reported by Kydland and Prescott (1982) and Hansen (1985).

This should not be too surprising. As the analysis above indicated, the analytical complexity of nonlinear models greatly exceeds that associated with linear stochastic models. The primitive stage of our research technology forced us to work with a rather rudimentary and rigid model. The introduction of an elastic labor supply, a nonlinear utility function, and increasing returns to scale in production are all elements of realism that may improve the performance of our model. Nevertheless, our study casts some doubt on the notion that, in one-dimensional capital-good models, chaos is a useful way to model the apparently self-sustained nature of the trade cycle. The highly nonlinear bell-shaped form for the policy function that is then necessary in order to
produce complex dynamics forces one to resort to rather unrealistic values of the parameters. Future research on models with a higher-dimensional state space may be more successful in this regard, since it appears that even slight departures from linearity may then produce strange attractors.

While still in its infancy, the study of nonlinearities in economic models is likely to provide insights into the forces behind observed economic fluctuations. In our model, we underlined the importance of intersectoral substitution effects (induced by different degrees of profitability in different sectors) as well as intertemporal substitution effects in determining factor allocation decisions, investment activities, and so on. Bypassing the nonlinearities with first-order approximations would have neglected the important contribution of these factors in amplifying and sustaining oscillations.

Appendix

A.1. Proof of Proposition 5

We only sketch the proof here since it is an adaptation of the main argument in Benhabib and Nishimura (1985, theorem 1). When \( k_i \neq \gamma \), the Euler equation induces a map \( F: \tilde{D} \rightarrow X \), defined as \( k_{i+1} = F(k_i, k_{i-1}) \). From this we may derive a dynamical system \( H: \tilde{D} \rightarrow D \):

\[
\begin{bmatrix}
k_{t+1} \\
y_{t+1}
\end{bmatrix} = \begin{bmatrix}
F(k_t, y_t) \\
k_t
\end{bmatrix} = H(k_t, y_t).
\]

If \( x(\delta), y(\delta) \) satisfies \( \tau_\delta(x(\delta)) = y(\delta), \tau_\delta(y(\delta)) = x(\delta) \), it must also satisfy \( x = F(y, x), y = F(x, y) \), or \([x, y] = H(y, x)\).

Now set \( m_t = (k_t, y_t) \). We have

\[
m_{t+2} = H(m_{t+1}) = H(H(m_t)) = H^2(m_t) = \begin{bmatrix}
F(F(k_t, y_t), k_t) \\
F(k_t, y_t)
\end{bmatrix},
\]

and \([x, y] = H^2(x, y)\), so that the period-two point is a fixed point for \( H^2 \). Let us compute the eigenvalues of \( DH^2 \) at \((x, y)\). Simple but tedious algebra shows that \( \lambda_1 \lambda_2 = \text{Det}(DH^2(x, y)) = \delta^{-2} \) and that \( \lambda_1 + \lambda_2 = \text{tr}(DH^2)(x, y) = [V_{22}^* + \delta^2V_{11}^* + \delta V_{12}^* + \delta V_{22}^* - \delta V_{12}^2 - \delta V_{22}^2] / \delta^2 V_{12}^* V_{12}^* \).

It is straightforward to show that the concavity of \( V \) implies that both roots are real. Without loss of generality, let us assume \( |\lambda_1| \leq |\lambda_2| \). From \( \lambda_1 \lambda_2 = \delta^{-2} \) we see that \( \lambda_1 \in (-\infty, -1) \) for all \( \delta \) in the relevant interval. In particular, if \( \lambda_2 \) happens to be equal to \(-1\) for some \( \delta \), then \( \lambda_1 = -\delta^{-2} \) for some
\( \delta \in (\delta^{--}, \delta^{++}) \). Furthermore,

(i) if \((1 + \delta^2) V_1^* V_{12} + [V_{22}^* V_{22} + \delta^2 V_{11}^* V_{12} + \delta(V_1^* V_{12}^2 - V_{12}^2) + \delta(V_{11}^* V_{22} - V_{12}^2)] = 0 \),
then \( \lambda_1 = -\delta^{--} \) and \( \lambda_2 = -1 \);

(ii) if \((1 + \delta^2) V_1^* V_{12} + [V_{22}^* V_{22} + \delta^2 V_{11}^* V_{11} + \delta(V_1^* V_{12}^2 - V_{12}^2) + \delta(V_{11}^* V_{22} - V_{12}^2)] > 0 \),
then \( \lambda_1 \in (-\infty, -1) \) and \( \lambda_2 \in (-1, 0) \);

(iii) if \((1 + \delta^2) V_1^* V_{12} + [V_{22}^* V_{22} + \delta^2 V_{11}^* V_{11} + \delta(V_1^* V_{12}^2 - V_{12}^2) + \delta(V_{11}^* V_{22} - V_{12}^2)] < 0 \),
then \( \lambda_1 \) and \( \lambda_2 \) both belong to \( (-\infty, -1) \).

From here on, the proof proceeds analogously to Theorem 1 in Benhabib and Nishimura (1985).

A.2. Proof of the corollary to Proposition 4

\( G(\delta) \), for our model, may be written as

\[
\alpha^2 (1 - \alpha)^2 \left( \frac{1 - y}{x - \gamma y} \right)^a \left( \frac{1 - x}{x - \gamma y} \right)^a \left( x - \gamma y \right)^{-1} \left( y - \gamma x \right)^{-1} \\
\times \left[ \left( \frac{x - \gamma}{1 - x} \right)^2 \left( \frac{y - \gamma}{1 - y} \right)^2 + \delta^2 + (1 + \delta)^2 \left( \frac{x - \gamma}{1 - x} \right) \left( \frac{y - \gamma}{1 - y} \right) \right].
\]

We only have to consider the portion within the square brackets, which is quadratic in \((x - \gamma)/(1 - x)\) and \((y - \gamma)/(1 - y)\). Simple calculations reveal that \( G(\delta) = 0 \) when either condition (i) or (ii) of the corollary are satisfied. Moreover, as the sign of \( G(\delta) \) changes when \( \delta \) moves through a zero, the argument of Proposition 4 can be applied.

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