On the Differentiability of the Policy Function*

Michele Boldrin

University of California, Los Angeles

and

Luigi Montrucchio

University of Torino, Torino, Italy

UCLA Dept. of Economics
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1 Introduction

In the tool kit of both applied and theoretical economics the basic model of intertemporal utility maximization has always played a special role, at least since the time of Ramsey. It is fair to say that it constitutes, as of today, the most often applied instruments of formal analysis both in empirically-oriented and theoretical areas.

While for most of the 1960s and 1970s the continuous time approach, with the associated Hamiltonian formalism, seemed to dominate research, in more recent years economists have concentrated their attention on the discrete-time setup and Dynamic Programming techniques are more often adopted.

From our standpoint, the major advantage of the Dynamic Programming approach, besides analytical and computational tractability, is the property that all the relevant information about the optimal solutions, their dynamic behavior and their comparative statics and dynamic properties, is summarized in a single mathematical object: the policy function τ_δ (see below for a formal definition). While the continuity properties of τ_δ are well-understood even in the most general framework, much less is known about its deeper structure (bi-furcational behavior, smoothness, monotonicity, etc.) Indeed, as the results of Boldrin-Montrucchio [1986a, b] (and Montrucchio [1988] for the continuous-time case) have shown, there are no restrictions one can impose on the dynamical properties of the policy function, at least for the general model. One particular issue upon which various researchers seem to have concentrated their attention is the C^1-differentiability of the optimal policy (see especially Lucas et. al. [1989, Chs. 5 and 6]). Indeed, to know that τ_δ is of class C^1 with respect to both the state variables and the parameters of the model would prove extremely useful in the study of its bifurcations as well as in proving comparative statics and dynamic properties. Furthermore, as recently pointed out in Kehoe et al. [1989], the C^1-differentiability of τ_δ (or, better, the almost equivalent C^2-differentiability of the value function W_δ, see Proposition **??** and **??** for details) is sufficient to guarantee the local uniqueness of equilibria in the standard, infinite horizon, neoclassical, general equilibrium model with finitely many agents and production. In this paper we address the question of the C^1-differentiability of τ_δ within a very general framework. Indeed, it is our perception that none of the assumptions we make can be significantly weakened without affecting the result. We provide some examples in this sense in Appendix B. A brief overview of the relevant literature on this topic seems to be in order here. The first paper addressing the issue is Araujo and Scheinkman [1977]: they show that when the global asymptotic turnpike property is satisfied (i.e., all optimal paths converge to a unique optimal stationary position) then one can invert a certain mapping (induced by the Eueler equations) acting on the Banach space ℓ^∞. An application of the implicit function theorem for Banach spaces then delivers the result. In

Benveniste and Scheinkman [1979] the C^1-differentiability of the value function is proved with respect to the states, but no consideration given to the C^1-smoothness of the policy. Moreover, the method used there does not appear to be extendible to the study of the C^1-differentiability of the value function with respect to parameters. Montrucchio [1987] studies the Lipschitz continuity of the policy function with respect to the state variables: he proves the property to be true under very mild conditions. The critical one, i.e., strong convexity of the underlying return function V, is retained here. Unfortunately, his method does not also extend to the study of the dependence of value and policy functions on the parameter vector. Two recent papers are more closely related to the present. Araujo [1988] considers the one-dimensional case: he assumes the policy is monotonically increasing and derives the C^1 property by showing it is such at any of its fixed points and then extending the result globally. The work by Santos [1989] addresses the very same general model considered here. He uses our same assumptions plus a hypothesis of global invertibility of the matrix $V_{12}(x, y)$ of mixed partial derivatives of the return function, which is instead dispensed with in the present work. The technique of proof is, nevertheless, quite different from ours: while we study the contraction property of a certain operator acting on a Banach space, he uses the theory of invariant manifolds (stable, unstable and center) as developed by dynamical system theorists (see especially Hirsch, Pugh and Shub [1977]. By proving that the graph of the function τ_δ can be identified, locally, with an invariant manifold, he is able to use the smoothness of the latter to derive that of the former. It is worth pointing out that Araujo [1988], Santos [1989], and the present paper have all been developed independently. Finally, we should mention the work of Blume, Easley and O'Hara [1982]: they study the stochastic version of the Dynamic Programming problem and prove that the value is C^2. Their result depends heavily on the properties of the stochastic framework and cannot be applied by any means to the deterministic one. The remainder of this article proceeds as follows. Section 2 illustrates the model and some of its fundamental properties. As they are fairly well known, they are stated without proof. Boldrin-Montrucchio [1989] can be consulted for the proofs. In Section 3 we state the problem, derive the operator which is the central object of study and show that τ_δ is C^1 when such an operator is a contraction. In Section 4 we prove that under our assumption the contraction property is realized. Section 5 contains brief final comments. The two Appendices are as follows: Appendix A shows that with some algebra, the same result applies with respect to a parameter vector. This enables us to keep the notation in the main body of the paper within reasonable bounds. Appendix B provides counterexamples to the C^1 property when some of our hypotheses are violated.

2 Model and Assumptions

2.1 Notation

All of our analysis will be conducted in $\mathbb{R}^n$. For vectors $x \in \mathbb{R}^n$, the norm $\|x\|$ is the Euclidean one. For matrices $A \in M(R^n)$ (the set of all $n \times n$ real matrices) we will adopt the "spectral-norm" $\|A\| = \rho_M(A^T A)$, where $\rho_M(\bullet)$ is the square root of the modulus of the largest (in modulus) eigenvalue of $A^T A$. We will also use (not as a norm) the minimum eigenvalue of a matrix which we denote $\rho_m(\bullet)$. When symmetric matrices are used the following convention is adopted: $A \geq B$ implies $A - B \geq 0$ and it means that $(A - B)$ is a positive definite matrix.

To indicate spaces of functions we wil use the standard C^j notation, $j = 0, 1, \cdots$, so $C^0(X; Y)$, for example, denotes the set of all continuous functions from X into Y. Range and domain will be reported only when necessary to avoid confusion. For functions $f\colon \mathbb{R}^n \to \mathbb{R}$ we use the prime-notation to denote the gradient-vector $f'(x)$ and the Hessian-matrix $f''(x)$. For functions: $\theta\colon \mathbb{R}^n \to \mathbb{R}^m$ the notation $D\theta(x)$, $D^2\theta(x)$ etc., will be used to remind us that they are (multi)linear mappings.

Finally, for functions $V\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $V_i(x)$, for $i = 1, 2$, will denote the gradients of first derivatives with respect to the ith variable and, analogously, V_{ij}, $i, j = 1, 2$, denote the matrices of second-order derivatives. For a set X we will denote with $\text{int}\,(X)$ its interior.

2.2 The Model

We will be studying the problem:

$$W_\delta(x) = \max \sum_{t=0}^{\infty} V(x_t, x_{t+1})\delta^t \quad \text{such that} \quad (x_t, x_{t+1}) \in D \qquad (P)$$
$$x_0 = x \quad \text{fixed in} \quad X$$
$$0 < \delta < 1$$

Under the assumptions: (A.1) $X \subset \mathbb{R}^n$ is compact and convex, with non-empty interior; $D \subset X \times X$ is also compact, convex, $\text{int}\,(D) \neq \phi$ and $\text{Proj}_1(D) = X$. The correspondence $\Gamma\colon X \to X$ defined as $\text{graph}\,(\Gamma) = D$ is continuous. (A.2) $V\colon D \to \mathbb{R}$ is of class C^2 on D. $V(x, y)$ is strongly concave, with negative-definite Hessian on D. $V(x, \bullet)$ is strongly concave for each $x \in X$, i.e., there exists a real number $\beta > 0$ such that $|z' V_{22}(x, y) z| \geq \beta \|z\|^2$, holds for all $x, y \in X$ and all z in $\mathbb{R}^n$. The existence of an optimal

solution to (P) under (A.1)–(A.2) has been extensively studied and it is a well-understood problem (see McKenzie [1986]). Let us recall that the value function $W_\delta(x)$, defined in (P) as dependent on the initial conditions, is the unique fixed point of the δ-contraction : $B\colon C^0(X;R) \to C^0(X;R)$:

$$(Bf)(x) = \max_{y\in\Gamma(x)} [V(x,y) + \delta f(y)] \tag{1}$$

where $\Gamma(x) = \{y \in X \quad \text{such that} \quad (x,y) \in D\}$. Therefore W_δ can be reached starting from *any* initial condition W_0, just by iterating the operator B to product the sequence of functions:

$$W_T(x) = \max_{y\in\Gamma(x)} [V(x,y) = \delta W_{T-1}(y)] \quad T = 1,2,3,\cdots \tag{2}$$

In fact $W_T \to W_\delta$ uniformly with T (see Boldrin-Montrucchio [1989] for a proof). This simple fact will be used extensively in this paper. To the sequence of "values" W_t induced by (**??**) we will associate another sequence of "policy" functions θ_T defined as the maps from X into X that satisfy:

$$V\left(x,\theta_T(x)\right) + \delta W_{T-1}\left(\theta_T(x)\right) = \max_{y\in\Gamma(x)} \left[V(x,y) + \delta W_{T-1}(y)\right]. \tag{3}$$

Remark 2.1. Assume that, in (**??**) above, W_0 had been chosen to be concave: the whole sequence of W_T will then be concave in force of (A.2). In fact $V(x,\bullet) + \delta W_{T-1}(\bullet)$ will be strictly concave for each $x \in X$, so that the function $\theta_T(x)$ in (**??**) is well defined and unique. It is also continuous, as it may be seen by applying the Maximum Lemma.

Remark 2.2. The fact that the value functions W_t converges formly to the fixed point W_δ of (**??**) also implies that the associated θ_T converges uniformly to a continuous function $\tau_\delta\colon X \to X$, which is defined implicitly by:

$$V\left(x,\tau_\delta(x)\right) + \delta W_\delta\left(\tau_\delta(x)\right) = \max_{y\in\Gamma(x)} \left[V(x,y) + \delta W_\delta(y)\right]. \tag{4}$$

For a proof of this fact one may consult Boldrin-Montrucchio [1989]. The function τ_δ defined in (**??**) is named the *optimal policy* for the infinite horizon problem (P). Our strategy can now be easily described. We want to find sufficient conditions on (P) that will enable us to show that the value function $W_\delta\colon X \to I\!R$, defined as the fixed point of B: $C^0(X;I\!R) \to C^0(X;I\!R)$:

$$W_\delta(x) = \max_{y\in\Gamma(x)} [V(x,y) + \delta W_\delta(y)] \tag{5}$$

is of class $C^2(X;R)$ when (A.1)–(A.2) and (A.3) (see below) are satisfied. It is intuitive, and it will be shown rigorously later on, that under our

conditions this is equivalent to the policy function τ_δ to be of class $C^1(X;X)$. Clearly this cannot be always the case if τ_δ is not "interior." We assume therefore: (A.3) If τ_δ solves (**??**) then $\tau_\delta(x) \in \text{int}\,(\Gamma(x))$ for all $x \in X$

and for all $\delta \in (0,1)$. In order to talk about C^2-differentiability of W_δ

one obviously needs to prove first that W_δ is at least C^1. This is the well-known result of Benveniste and Scheinkman [1979], which in fact holds under conditions much weaker than (A.1)–(A.3). We report it here without proof.

Proposition 2.1. Under the maintained hypothesis the value function $W_\delta(x)$ is of class C^1 on X and $W'_\delta(x) = V_1\left(x, \tau_\delta(x)\right)$ for all $x \in X$.

Remark 2.3. If $\tau_\delta(x)$ satisfies (A.3) and V satisfies (A.2), then $\tau_\delta(x)$ is Lipschitz continuous and $V_1\left(x, \tau_\delta(x)\right)$ is also Lipschitz continuous (see Montruchhio [1987]). This implies, from Proposition **??** that $W'_\delta(x)$ is also Lipschitz continuous.

Remark 2.4. The method of Benveniste and Scneinkman [1979] unfortunately cannot be used to show that $W_\delta(x,\omega)$ is also C^1 with respect to the vector of parameters ω. We provide an alterante proof below using a method that extends to the parameters as well.

We will now address formally the relation between $W_\delta \in C^2$ and $\tau_\delta \in C^1$.

Proposition 2.2. If V is of class C^2 and τ_δ satisfies (A.3), then $\tau_\delta(x)$ differentiable at x_0 implies that $W_\delta(x)$ is twice differentiable at x_0.

Proof. From Proposition **??**: $W'_\delta(x) = V_1\left(x, \tau_\delta(x)\right)$ for all x in a neighborhood of x_0, because τ_δ is continuous. Then, if $\tau_\delta(x)$ is differentiable at x_0, $W'_\delta(x)$ will be differentiable at x_0 and:

$$W''_\delta(x_0) = V_{11}\left(x_0, \tau_\delta(x_0)\right) + V_{12}\left(x_0, \tau_\delta(x_0)\right) D\tau_\delta(x_0).$$

□

While Proposition **??** has been proven under a hypothesis much weaker than (A.2) (i.e., strong concavity is not used) we need to introduce strong concavity to prove the reverse implications.

Proposition 2.3. Under the hypotheses (A.1), (A.3) and the weak form of (A.2) given by: $-V$ is of class C^2 and $V(x,\bullet)$ is β-concave, then: if $W_\delta(x)$ is of class C^2 around $x_0 \in X$ then $\tau_\delta(x)$ is of class C^1 around x_0.

Proof. From the Bellman equation we have:

$$V_2\left(x, \tau_\delta(x)\right) + \delta W'_\delta\left(\tau_\delta(x)\right) = 0$$

which defines τ_δ implicitly for x in a neighborhood of x_0. The implicit function theorem shows that $\tau_\delta(x)$ is differentiable at x_0 if $\left[V_{22}\left(x, \tau_\delta(x)\right) + \delta W''_\delta\left(\tau_\delta(x)\right)\right]$ is invertible at x_0. β-concavity of V implies that all the eigenvalues of $V_{22}\left(x_0, \tau_\delta, (x_0)\right)$ are strictly negative and the matrix above is therefore invertible. One may easily compute that

$$D\tau_\delta(x_0) = -\left[V_{22}\left(x_0, \tau_\delta, (x_0)\right) + \delta W''_\delta\left(\tau_\delta(x_0)\right)\right]^{-1} V_{21}\left(x_0, \tau_\delta(x_0)\right).$$

□

Remark 2.5. Some of the hypotheses used in Proposition **??** can be weakened if τ_δ is locally invertible (see Boldrin-Montrucchio [1989]). The hypothesis of β-concavity on $V(x, \bullet)$ cannot be eliminated: when it is violated one can give examples where V and W_δ are of class C^2 but τ_δ is not differentiable (see Appendix B of this paper).

Finally we have:

Corollary 2.1. Under the maintained hypothesis if $W_\delta(x)$ is of class C^2 at $x \in X$ one has:

$$W''_\delta(x) = V_{11}\left(x, \tau_\delta(x)\right) - V_{12}\left(x, \tau_\delta(x)\right)\left[V_{22}\left(x, \tau_\delta(x)\right) + \delta W''_\delta\left(\tau_\delta(x)\right)\right]^{-1} V_{21}\left(x, \tau_\delta(x)\right).$$

3 A Preliminary Theorem

Suppose now that an initial function W_0 is chosen such that all the iterates W_T induced by (**??**) are of class C^2 and the associated θ_T are interior to D. A simple convergence argument will show that when the continuous functions W''_T converge, they converge to W''_δ, which is continuous itself. This basic intuition will be exploited in Section 4. We begin by showing that an appropriate initial condition W_0 can be chosen.

LEMMA 3.1. Let $W_\delta \colon X \to I\!R$ be as defined in (**??**) under (A.1)–(A.3). Then there exists $W_0 \in C^2(X; I\!R)$ close enough to W_δ in the C^0-topology so that all the θ_T induced by, (**??**) are interior.

Proof. Denote with $\widetilde{C}^0(D; I\!R)$ the space of all continuous, strictly concave functions from D into the real numbers. Then the map $J \colon \widetilde{C}^0(D; I\!R) \to C^0(X; X)$ defined as:

$$J(U) = \theta(x) = \text{Arg Max}\left[U(x, y); y \in \Gamma(x)\right]$$

is continuous, (see, e.g., Boldrin and Montrucchio [1989], Ch. 2). As τ_δ is interior and it satisfies $J(V + \delta W_\delta) = \tau_\delta$, and X is compact, it is possible to find a $\sigma > 0$ such that:

$$\theta(x) = J(V + \delta f)$$

is also interior for any continuous and concave function $f\colon X \to I\!R$, such that $\|f - W_\delta\| \leq \sigma$. The density of $\widetilde{C}^k(X;R)$ in $\widetilde{C}^o(X;R)$ for $k \geq 1$, guarantees that we can pick a $W_0 \in \widetilde{C}^2(X;R)$ such that $\|W_0 - W_\delta\| \leq \sigma$. Therefore by repeated applications of the δ-contraction operator B defined in (**??**) we obtain $\|W_T - W_\delta\| \leq \delta^T \sigma < \sigma$, $T = 1, 2, \ldots$. As $\theta_T = J(V + \delta W_T)$ we conclude that it is interior for all $T = 1, 2, \ldots$. □

LEMMA 3.2. Under (A.1)–(A.3) for W_0 chosen so as to satisfy Lemma (**??**), then all the W_T defined in (**??**) belongs to $\widetilde{C}^2(X;R)$ and all the θ_T defined in (**??**) belongs to $C^1(X;R)$. Moreover, the first derivatives of the θ_T's and the second derivatives of the W_T's (as linear maps from $I\!R^n$ into itself) can be computed recursively as

$$D\theta_T(x) = -L_{T-1}^{-1}\left(x, \theta_T(x)\right) \bullet V_{21}\left(x, \theta_T(x)\right) \tag{6}$$

and

$$W_T''(x) = V_{11}\left(x, \theta_T(x)\right) - V_{12}\left(x, \theta_T(x)\right) L_{T-1}^{-1}\left(x, \theta_T(x)\right) V_{21}\left(x, \theta_T(x)\right) \tag{7}$$

where

$$L_{T-1}(x, y) = V_{22}(x, y) + \delta W_{T-1}''(y).$$

Proof. It is easily obtained by induction. We have assumed $W_0 \in C^2$. Let us show that when W_{T-1} is C^2 then W_T is also C^2 and θ_T is C^1. From (**??**)–(**??**), the (strong) concavity of $V(x, y) + \delta W_{T-1}(y)$ and the interiority of $\theta_T(x)$ it follows that the first order condition:

$$V_2\left(x, \theta_T(x)\right) + \delta W_{T-1}'\left(\theta_T(x)\right) = 0 \tag{8}$$

must be satisfied. That θ_T is C^1 follows then from the implicit function theorem. In fact (**??**) is readily obtained by applying such theorem to the implicit function (**??**) and noticing that $V_{22}\left(x, \theta_T(x)\right) + \delta W_{T-1}''\left(\theta_T(x)\right)$ is invertible in force of the strong concavity of $V(x, \bullet)$, (A.2)). Furthermore:

$$W_T(x) = V\left(x, \theta_T(x)\right) + \delta W_{T-1}\left(\theta_T(x)\right),$$

so that W_T is C^2 and: $W_T'(x) = V_1\left(x, \theta_T(x)\right)$. The main Hessian matrix $W_T''(x)$ can again be computed by differentiating the equality above and using (**??**). □

It is now simple to prove:

Theorem 3.1. Under the assumptions (A.1)–(A.3) if the sequence $\{W_T''\}_{T=0}^{\infty}$ defined in (**??**) converges uniformly one has: $C^0 - \lim_{T\to\infty} W_T'' = W_\delta''$ and $C^0 - \lim_{T\to\infty} D\theta_T = D\tau_\delta$. Therefore $W_\delta \in C^2$ and $\tau_\delta \in C^1$.

Proof. We know that $W_T \to W_\delta$ uniformly with $T \to \infty$. We also that $W_T'(x) = V_1(x, \theta_T(x))$ for all T. Therefore:

$$\lim_{T\to\infty} W_T'(x) = \lim_{T\to\infty} V_1(x, \theta_T(x)) = V_1(x, \tau_\delta(x)) = \psi(x).$$

A well known approximation theorem (see Dieudonné [1969], #8.6.3) then implies that $\Psi(x) = W_\delta'(x) = \lim W_T'(x) = V_1(x, \tau_\delta(x))$. (This is an alternative proof of the result of Benveniste-Scheinkman [1979].) We can now proceed with the second order derivatives. As $\{W_T''\}_{T=0}^{\infty}$ is converging uniformly we obviously have: $W_T' \to W_\delta'$ uniformly with T, and also:

$$W_T'' \to \psi(x) \in C^0(X; M(\mathbb{R}^n))$$

uniformly as $T \to \infty$. Again the same approximation result quoted above implies:

$$\psi(x) = \frac{d\Psi(x)}{dx} = W_\delta''(x) = C^0 - \lim W_T''(x).$$

This implies that $\lim_{T\to\infty} W_T''$ exists, is continuous and equal to W_δ''. The result for the policy functions θ_T and τ_δ is then a straightforward consequence of (**??**). □

4 A More General Approach

The theorem proved in Section 3 is our starting pont. We have shown that we need to prove that the sequence $\{W_T''\}_{T=0}^{\infty}$ defined in (**??**) converges uniformly on X in order to guarantee that $W_\delta \in C^2$ and $\tau_\delta \in C^1$. We will now try to understand if this is always the case under the sole hypotheses (A.1)–(A.3) or not. Let $C^0(X; S^-(\mathbb{R}^n))$ denote the space of all continuous functions ϕ from X into the set of all $n \times n$, symmetric, negative-definite real matrices. The topology on $C^0(X; S^-(\mathbb{R}^n))$ is, once again, the one induced by the sup norm. Then equation (**??**) of Section 3 can be interpreted as defining the operator $f_T \colon C^0(X; S^-(\mathbb{R}^n)) \to C^0(X; S^-(\mathbb{R}^n))$ as

$$(f_T\phi)(x) = V_{11}(x, \theta_T(x)) - V_{12}(x, \theta_T(x)) L_\phi^{-1}(x, \theta_T(x)) V_{21}(x, \theta_T(x)) \tag{9}$$

where: $L_\phi(x, y) = V_{22}(x, y) + \delta\phi(y)$. The sequence $\{W_T''\}_{T=0}^{\infty}$ can therefore be expressed as $W_T'' = f_T(W_{T-1}'')$. By analogy we can define also the operator $f_\infty \colon C^0(X; S^-(\mathbb{R}^n)) \to C^0(X; S^-(\mathbb{R}^n))$ as:

$$(f_\infty\phi)(x) = V_{11}(x, \tau_\delta(x)) - V_{12}(x, \tau_\delta(x)) L_\phi^{-1}(x, \tau_\delta(x)) V_{21}(x, \tau_\delta(x)). \tag{10}$$

LEMMA 4.1. The operators f_t and f_∞ are well defined from $C^0\left(X; S^-(I\!R^n)\right)$ into itself.

Proof. We consider only f_∞, a similar argument holds for f_T. Write $f_\infty(\phi)$ as: $V_{11} - V_{12}(V_{22} + \delta\phi)V_{21}$ to economize on notation. Hypothesis (A.3) implies $V_{22} \leq -\beta I$ and so $V_{22} + \delta\phi \leq \beta I$ for any $\phi \in S^-(I\!R^n)$. $V_{22} + \delta\phi$ is therefore invertible and $f_\infty(\phi)$ is well defined. We need to show that $(f_\infty\phi)(x) \in C^0\left(X; S^-(I\!R^n)\right)$. Notice that $V_{11} - V_{12}V_{22}^{-1}V_{21} \leq 0$ follows from the assumption of concavity of V. Hence, for $\phi \in S^-(I\!R^n)$, we have: $V_{22} + \delta\phi \leq V_{22} \leq -\beta I$, which implies $V_{22}^{-1} \leq (V_{22} + \delta\phi)^{-1}$ and, in turn, $V_{12}V_{22}^{-1}V_{21} \leq V_{12}(V_{22} + \delta\phi)^{-1}V_{21}$. Therefore:

$$0 \geq V_{11} - V_{12}V_{22}^{-1}V_{21} \geq V_{11} - V_{12}(V_{22} + \delta\phi)^{-1}V_{21} = f_\infty(\phi)$$

□

LEMMA 4.2. Let $S \subset B$ be a closed subset of a Banach space B and $F_\infty \colon S \to S$ a mapping which is a contraction over S. Consider a sequence $\{f_n\}_{n=1}^\infty$ of maps $f_n \colon S \to S$ that converges uniformly to f_∞ on S. Then the iterative system: $x_n = f_n(x_{n-1})$ converges, for any given $x_0 \in S$, to the fixed point x_∞ of f_∞.

Proof. Let $\alpha \in [0, 1)$ be the contractive factor for f_∞, i.e., $\|f_\infty(x_1) - f_\infty(x_2)\| \leq \alpha\|x_1 - x_2\| \,\forall\, x_1, x_2 \in S$. Then for given $\epsilon > 0$ we can find $N = N(\epsilon)$ such that: $\|f_n(x) - f_\infty(x)\| \leq \epsilon(1-\alpha)/2$ holds for all $n \geq N$ and all $x \in S$. Therefore: $\|x_{N+1} - x_\infty\| = \|f_{N+1}(x_N) - f_\infty(x_\infty)\| \leq \|f_{N+1}(x_N) - f_\infty(x_N) + f_\infty(x_N) - f_\infty(x_\infty)\| \leq \epsilon(1-\alpha)/2 + \alpha\|x_N - x_\infty\|$. From the latter, by iteration of f_N, it is easy to see that:

$$\|x_{N+n} - x_\infty\| \leq \frac{\epsilon(1-\alpha)}{2}(1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}) + \alpha^n\|x_N - x_\infty\| \leq \frac{\epsilon}{2} + \alpha^n\|x_N - x_\infty\|.$$

As $\alpha^n\|x_N - x_\infty\| \to 0$ when $n \to \infty$, there exists an $n_0(\epsilon)$ such that $\alpha^n\|x_N - x_\infty\| \leq \epsilon/2$ for $n \geq n_0(\epsilon)$. In conclusion for any $\epsilon > 0$ we have that for $m \geq N(\epsilon) + n_0(\epsilon)$:

$$\|x_N - x_\infty\| \leq \|x_{N+(m-N)} - x_\infty\| \leq \frac{\epsilon}{2} + \alpha^{m-N}\|x_N - x_\infty\| \leq \epsilon.$$

□

LEMMA 4.3. Let $\{f_n\}_{n=1}^\infty$ and f_∞ satisfy the same hypotheses as in Lemma **??**. Assume furthermore that: (a) f_∞ is uniformly continuous on S; and (b) there exists an integer $N \geq 1$ such that f_∞^N is a contraction, i.e., $\|f_\infty^N(x_1) - f_\infty^N(x_2)\| \leq \alpha\|x_1 - x_2\|$ for $\alpha \in [0, 1)$ where f_∞^N denotes the Nth iterate of f_∞. Under these conditions the sequence $x_n - f_n(x_{n-1})$ converges uniformly to the unique fixed point x_∞ of f_∞.

Proof. That f_∞ still has a unique fixed point is a well known property (see, e.g., Kolmogorov-Fomin [1970, pp. 70]). We want to use Lemma **??** in order to show that the iterative system $x_n = (f_n \bullet f_{n-1} \bullet \cdots \bullet f_{n-N+1})(x_{n-N})$, (here the symbol $\bullet$ denotes composition of mappings) with initial conditions $[x_0, f_1(x_0), f_2 \bullet f_1(x_0), \dots, f_{N-1} \bullet f_{N-2} \bullet \cdots \bullet f_1(x_0)]$, satisfies $\lim_{n\to\infty} x_n = x_\infty$. In order to do so we need to prove that, for any integer $k \geq 1$, the sequence of maps $\{f_n \bullet f_{n-1} \bullet \cdots \bullet f_{n-k+1}\}_{n=1}^{\infty}$ converges uniformly to f_∞^k, under the hypothesis of this lemma. As the above property is satisfied by assumption when $k = 1$, let us show that, if it is satisfied for some k, then it holds also for $k+1$. Indeed,

$$\begin{aligned}
&\|f_n \bullet (f_{n-1} \bullet \cdots \bullet f_{n-k})(x) - f_\infty(f_\infty^k)(x)\| \\
&\qquad = \|f_n \bullet (f_{n-1} \bullet \cdots \bullet f_{n-k}(x) - f_\infty(f_{n-1} \bullet \cdots \bullet f_{n-k})(x) \\
&\qquad\quad + f_\infty(f_{n-1} \bullet \cdots \bullet f_{n-k})(x) - f_\infty(f_\infty^k)(x)\| \\
&\qquad \leq \|f_n \bullet (f_{n-1} \bullet \cdots \bullet f_{n-k}(x) - f_\infty(f_{n-1} \bullet \cdots \bullet f_{n-k}(x)\| \\
&\qquad\quad + \|f_\infty \bullet (f_{n-1} \bullet \cdots f_{n-k}(x) - f_\infty(f_\infty^k)(x)\|.
\end{aligned}$$

From the uniform continuity of f_∞ and the induction hypothesis we know that for any $\epsilon > 0$ there exists n_0, such that for $n \geq n_0$ the last two terms of the above inequality are less than $\epsilon/2$. Therefore: $\|f_n \bullet \cdots \bullet f_{n-k}(x) - f_\infty^{k+1}(x)\| \leq \epsilon$ which proves the uniform convergence for all k. Choosing $k = N$ the hypothesis of Lemma **??** will be satisfied and the result follows. □

Now we can prove:

Theorem 4.1. The value function W_δ defined in (**??**) is of class C^2 if there exists an integer N such that the Nth iterate of f_∞ defined in (**??**) is a contraction mapping on $C^0\left(X; S^-(\mathbb{R}^n)\right)$.

Proof. We need only to apply Lemma **??** to the iterative system $W_T'' = f_T(W_{T-1}'')$ defined in (**??**) and to f_∞ as defined in (**??**). Recall that $\theta_T \to \tau_\delta$ uniformly with T and that X is compact: therefore $f_T \to f_\infty$ uniformly. We need only to check that f_∞ is uniformly continuous on $C^0\left(X; S^-(\mathbb{R}^n)\right)$. For any ϕ_1 and ϕ_2 in $C^0\left(X; S^-(\mathbb{R}^N)\right)$ we can compute:

$$\begin{aligned}
&(f_\infty\phi_1)(x) - (f_\infty\phi_2)(x) \\
&\qquad = \delta V_{12}[V_{22} + \delta\phi_2]^{-1}\left[\phi_1\left(\tau(x)\right) - \phi_2\left(\tau(x)\right)\right] \bullet [V_{22} + \delta\phi_1]^{-1}V_{21}.
\end{aligned}$$

Define $L = \sup \|V_{12}(x,y)\|$, $(x,y) \in D$, taking norms and rearranging the latter equation we have:

$$\|f_\infty(\phi_1) - f_\infty(\phi_2)\| \leq \delta\left(\frac{L}{\beta}\right)^2 \|\phi_1 - \phi_2\|.$$

This shows that f_∞ is Lipschitz and, therefore, uniformly continuous. By Lemma **??** this implies that W''_T converges uniformly to the unique fixed point of f_∞ when the latter satisfies the assumptions of this theorem. By the same argument used in the proof of Theorem **??** this will imply that $W_\delta \in C^2$. □

The theorem just proved reduces our problem to the contractivity of some finite, iteration of the operator f_∞. The strong concavity hypothesis (A.2) implies that there exists a real number $\bar{\alpha} > 0$ such that for all $0 \leq \alpha \leq \bar{\alpha}$, all $h \in I\!R^n$ and all $(x, y) \in D$ the following inequality holds:

$$h'[V_{11} - V_{12}V_{22}^{-1}V_{21}]h \leq -\alpha\|h\|^2 \tag{11}$$

The latter implies, in particular, that for any real number $\mu \geq 0$ we have:

$$-\bar{\alpha}I \geq V_{11} - V_{12}V_{22}^{-1}V_{21} \geq V_{11} - V_{12}(V_{22} - \mu I)^{-1}V_{21}, \tag{12}$$

where I is the identity matrix and we remind the reader that for symmetric matrices A, B the notation $A \geq B$ means $A - B$ is positive definite. From (**??**) it is easy to define the continuous real valued function $\psi\colon I\!R_+ \to I\!R_+$ as:

$$\psi(\mu) = \sup\{\alpha \geq 0 \quad \text{such that (**??**) holds for all} \quad h \in I\!R^n\} \tag{13}$$

where

$$h'\left[V_{11} - V_{12}(V_{22} - \mu I)^{-1}V_{21}\right]h \leq -\alpha\|h\|^2. \tag{14}$$

Remark 4.1. The function ψ was first introduced in Montrucchio [1987] and named "the ψ-transform" (of the concave real valued function V). The reader is referred to that paper and to Bodrin-Montrucchio [1989] for a proper account of the properties of ψ. Here we recall only that: $\psi(0) = \bar{\alpha}$ and that ψ is increasing, continuous, concave and bounded from above by $\rho_m(V_{11})$. This implies, in turn, that ψ certainly has one and only one strictly positive fixed point whenever $\bar{\alpha} > 0$.

For given $\delta \in [0, 1)$ consider now the equality:

$$\psi(\delta\alpha) = \alpha. \tag{15}$$

This also has one and only one solution $\alpha^*(\delta) \geq \bar{\alpha}$ which satisfies $\alpha^*(0) = \bar{\alpha}$ and is increasing in δ.

Definition 4.1. For any $\alpha > 0$ let $S^-(\alpha) \subset S^-(I\!R^n)$ be the closed subset of those $n \times n$ matrices A such that $h'Ah \leq -\alpha\|h\|^2$ holds for all $h \in I\!R^n$.

LEMMA 4.4. Under (A.1),(A.2), (A.3), and for given $\delta \in (0, 1)$ the operator f_∞ maps the space $C^0(X; S^-(\alpha))$ into itself for any given $\alpha \in [0, \alpha^*(\delta)]$.

Proof. Let δ be given in $(0,1)$ and $\alpha^*(\delta)$ be defined as in (**??**) above. Notice that $\psi(\delta\alpha) \geq \alpha > 0$ holds for our choice of α. Then $\phi \in C^0(X; S^-(\alpha))$ implies that: $\delta\phi \geq -\delta\alpha I$, hence $(V_{22}+\delta\phi)^{-1} \geq (V_{22}-\delta\alpha I)^{-1}$ and furthermore:

$$V_{11} - V_{12}(V_{22}+\delta\phi)^{-1}V_{21} \leq V_{11} - V_{12}(V_{22}-\delta\alpha I)^{-1}V_{21}$$

as V_{11} is negative definite. The left hand side of the last inequality is $f_\infty(\phi)$ and so we can conclude that:

$$f_\infty(\phi) \leq \psi(\delta\alpha)I \leq -\alpha I$$

in force of (**??**), (**??**) and the initial observation. □

LEMMA 4.5. Under (A.1), (A.2), and (A.3), and for given $\delta \in (0,1)$ the norm of:

$$L_\phi^{-1}(x) = [V_{22}(x, \tau(x)) + \delta\phi(\tau(x))]^{-1}$$

satisfies: $\|L_\phi^{-1}(x)\| \leq [\beta + \delta\alpha^*(\delta)]^{-1}$ for all $\phi \in C^0(X; S^-(\alpha^*(\delta)))$.

Proof. It follows at once from the Lemma above and the following observation:

$$\|L_\phi^{-1}(x)\| = \rho_M\left(L_\phi^{-1}(x)\right) = [\rho_m(L_\phi(x))]^{-1}.$$

On the other hand, $|z'L_\phi(x)z| \geq [\beta + \delta\alpha^*(\delta)]\|z\|^2$ for all $z \in I\!R^n$ because of Lemma **??**. The latter inequality says that the minimum eigenvalue of $L_\phi(x)$ is, in modulus, larger than $\beta + \delta\alpha^*(\delta)$ from which the result follows. □

LEMMA 4.6. Under the hypothesis (A.1), (A.2) and (A.3), the operator f_∞ is a contraction from the closed subspace $C^0(X; S^-(\alpha^*(\delta)))$ into itself for all $\delta \in (0,1)$ that satisfy the relation:

$$\frac{\delta L^2}{[\beta + \delta\alpha^*(\delta)]^2} < 1. \tag{16}$$

Proof. With the same technique used in the proof of Theorem **??** one can compute that, for any pair ϕ_1, ϕ_2 in $C^0(X; S^-(\alpha^*(\delta)))$ we have:

$$\|f_\infty(\phi_1) - f_\infty(\phi_2)\| \leq \frac{\delta L^2}{[\beta + \delta\alpha^*(\delta)]^2}\,\|\phi_1 - \phi_2\|.$$

□

All this can no be properly summarized in

Theorem 4.2. Under the hypothesis (A.1), (A.2), and (A.3) the value function $W_\delta\colon X \to I\!R$ defined in (P) is of class C^2 for all the values of $\delta \in [0,1)$ such that $\delta L^2 < [\beta + \delta\alpha^*(\delta)]^2$.

Proof. It follows from Lemma **??** and Theorem **??**. □

TO BE COMPLETED

5 Conclusions

TO BE COMPLETED

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A Dependence on Parameters

The problem in this case is:

$$(P\omega) \qquad W_\delta(x;w) = \max \sum_{t=0}^{\infty} V(x_t, x_{t+1}; \omega)\delta^t \quad \text{such that} \quad (x_t, x_{t+1}) \in D$$

under:

(A.1ω) $\omega \in \Omega \subset I\!R^m$, convex and compact. (A.2ω) $\widetilde{D} = D \times \Omega \subset I\!R^{2^n} \times I\!R^m$, where D is as in (A.1) of Section 2, and $\Gamma_\omega \colon X \to X$ is continuous for all $\omega \in \Omega$. (A.3ω) $V \colon \widetilde{D} \to I\!R$ is of class C^2 on $\widetilde{D}$. $V(\bullet, \bullet; \omega)$ is strongly concave (as defined in (A.2)) for every $\omega \in \Omega$. Also: $V(x, \bullet; \omega)$ is β-concave for every $X \in X$, $\omega \in \Omega$.

Remark A.1. In certain cases (welfare weights) we may add the hypothesis that V is indeed convex in ω.

The Bellman operator now maps $C^0(X \times \Omega; I\!R)$ into itself, i.e.,

$$(A.1) \qquad (Bf)(x;\omega) = \max_{y \in \Gamma(x)} \{V(x, y; \omega) + \delta f(y; \omega)\}.$$

For any $\omega \in \Omega$ Lemma **??** still holds, i.e., we can pick $W_0(x;\omega)$ such that the sequence $W_T(\bullet, \omega) \to W_\delta(\bullet, \omega)$ uniformly and maintaining all the $\theta_T(\bullet; \omega)$ interior. It is obvious that the fixed point W_δ of B is still continuous over $X \times \Omega$; the same is true for τ_δ which is now defined as:

$$W_\delta(x;\omega) = V\left(x, \tau(x;\omega); \omega\right) + \delta W_\delta\left(\tau(x;\omega); \omega\right).$$

Obviously $\theta_T(\bullet; w) \to \tau_\delta(\bullet; \omega)$ uniformly. The interiority hypothesis is now: (A.4ω) For any $\omega \in \Omega$, for all $x \in X$ and all $\delta \in (0,1)$: $\tau_\delta(x;\omega) \in \text{int}\ (\Gamma_\omega(x))$. Retaining notation from the main text we define:

$$L_{T-1}^{-1}(x, y; \omega) = [V_{22}(x, y; \omega) + \delta D_{11} W_{T-1}(y; \omega)]^{-1}.$$

Some tedious algebra (given at the end of this Appendix) gives the following representation for the first and second order derivatives of the $\{\theta_T, W_T\}_{T=0}^{\infty}$. (N.B.: The V_{ij} are here evaluated at $(x, \theta_T(x;\omega); \omega)$

whereas the $D_{ij}W_{T-1}$ are evaluated at $(\theta_T(x;\omega);\omega)$.

(A.2)
$$D\theta_T(x;\omega) = [D_1\theta_1(x;\omega); D_2\theta_T(x;\omega)] = -L_{T-1}^{-1}\left[V_{21}; V_{23} + \delta D_{12}W_{T-1}\right]$$

(A.3)
$$D_1W_T(x;\omega) = V_1\left(x, \theta_T(x,\omega);\omega\right)$$

(A.4)
$$D_2W_T(x;\omega) = V_3\left(x, \theta_T(\omega,\omega);\omega\right) + \delta D_2W_{T-1}\left(\theta_T(x;\omega);\omega\right)$$

(A.5)
$$D_{11}W_T(x;\omega) = V_{11} - V_{12}L_{T-1}^{-1}V_{21}$$

(A.6)
$$D_{12}W_T(x;\omega) = V_{13} - V_{12}L_{T-1}^{-1}(V_{23} + \delta D_{12}W_{T-1})$$

(A.7)
$$D_{21}W_T(x;\omega) = V_{31} - (V_{32} + \delta D_{21}W_{T-1})L_{T-1}^{-1}V_{21}$$

(A.8)
$$D_{22}W_T(s;\omega) = V_{33} + \delta D_{22}W_{T-1} - (V_{32} + \delta D_{21}W_{T-1})L_{T-1}^{-1}(V_{23} + \delta D_{12}W_{T-1})$$

To begin with we can establish:

LEMMA A.1. Under our assumptions (in fact much less) the value function is C^1 with respect to both x and ω.

Proof. From (**??**) we can, in analogy with (**??**) of Section 4, define the sequence of operator $g_T\colon C^0(X\times\Omega;\mathbb{R}^m) \to C^0(X\times\Omega;\mathbb{R}^m)$ as

$$(g_T\psi)(x,\omega) = V_3\left(x, \theta_T(x,\omega);\omega\right) + \delta\psi\left(\theta_T(x,\omega);\omega\right)$$

and their uniform limit $g_\infty\colon C^0(X\times\Omega;\mathbb{R}^m) \to C^0(X\times\Omega;\mathbb{R}^m)$ as

$$(g_\infty\psi)(x,\omega) = V_3\left(x, \tau_\delta(x,\omega);\omega\right) + \delta\psi\left(\tau_\delta(x,\omega);\omega\right).$$

That they are both well defined from $C^0(X\times\Omega;\mathbb{R}^m)$ into itself and that g_T converges uniformly to g_∞ it is easy to check. Now $C^0(X\times\Omega;\mathbb{R}^m)$ is also a Banach space in the sup norm and it is immediate to see that g_∞ is a δ-contraction on it. Using Lemma **??** we can deduce that D_2W_T will converge to the unique fixed point of g_∞. The same approximation theorem used in Theorem **??** above gives the desired conclusion. That W_δ is C^1 with respect to x has already been proved there. □

"Historical Remark": From conversations with José Scheinkman we discovered that he has been using (for at least ten years) the very same argument to prove this result during his class lectures at the University of Chicago. In fact, he pointed out to us that also in this instance the limit-operator argument was much more precise, neat and powerful than a recursive argument (reduce D_2W_T to $\sum_{i=0}^{T-1} V_3\left(X, \theta_{T-1}(x,\omega),\omega\right)\delta^i + \delta^T D_2 W_{T-1}$ and then take limits as $T \to \infty$) we had been using before. The right hand sides of (**??**)–(**??**) can be shown to converge (with $T \to \infty$) to four matrix-operators for the very same reasons for which $f_T \to f_\infty$ in Section 4. As (**??**) and **??**) are clearly one the transpose of the other we will concern ourselves with (**??**) only. Now from (**??**) we can derive the well known: $f_\infty \colon C^0\left(X \times \Omega; S^-\left(\alpha^*(\delta)\right)\right) \to C^0\left(X \times \Omega; S^-\left(\alpha^*(\delta)\right)\right)$. Using (**??**) we can also prove:

LEMMA A.2. Assume $\phi \in C^0\left(X; S^-\left(\alpha^*(\delta)\right)\right)$ for given $\delta \in [0,1)$. Define with: $f_\infty^{12} \colon C^0\left(X \times \Omega; M(n \times m)\right) \to C^0\left(X \times \Omega; M(n \times m)\right)$ (where $M(n \times m)$ is the space of all the $n \times m$ real matrices) the operator:

$$\begin{aligned}(f_\infty^{12} M)(x,\omega) = V_{13}\left(x, \tau_\delta(x,\omega);\omega\right) - V_{12}\left(x, \tau_\delta(x,\omega);\omega\right) L_\phi^{-1}(x,\omega) \\ \cdot \left[V_{23}\left(x, \tau_\delta(x,\omega);\omega\right) + \delta M\left(\tau_\delta(x,\omega),\omega\right)\right].\end{aligned}$$

The sequence $\{M_\phi^T\}_{T=0}^\infty$ in $C^0\left(X \times \Omega; M(n \times m)\right)$ obtained by iterating f_∞^{12} converges to a unique $M_\phi \in C^0\left(X \times \Omega; M(N \times m)\right)$ whenever the discount factor satisfies: $\delta L < \beta + \delta\alpha^*(\delta)$.

Proof. Given an initial condition $W_0 \in C^2(X \times \Omega; I\!R)$ the Tth component of the sequence M_ϕ^T can be written as:

$$M_\phi^T = \sum_{i=0}^{T-1} \left(-\delta V_{12} L_\phi^{-1}\right)^i (V_{13} - V_{12} L_\phi^{-1} V_{23}) \pm \left(\delta V_{12} L_\phi^{-1}\right)^T D_{12} W_0$$

(everything is obviously evaluated at (x,ω) or $(x, \tau(x,\omega);\omega)$ (the algebra is again at the end of the Appendix). By the usual technique we can estimate the largest eigenvalue of $\delta V_{12} L_\phi^{-1}$ to be bounded, in modulus, by $\delta L[\beta + \delta\alpha^*(\delta)]^{-1}$. The conclusion then follows. □

As an immediate consequence we have:

Corollary A.1. Assume $D_{11}W_\delta \in C_0\left(X \times \Omega; S^-\left(\alpha^*(\delta)\right)\right)$ exists for given δ. Then also $D_{12}W_\delta(x,\omega)$ (and its transpose $D_{21}W_\delta(x,\omega)$ exist under the conditions of Lemma **??** as continuous functions from $X \times \Omega$ into $M(n \times m)$. More precisely we have:

$$D_{12}W_\delta = \left[I + \delta V_{12}(V_{22} + \delta D_{11}W_\delta)^{-1}\right]^{-1}\left[V_{13} - V_{12}(V_{22} + \delta D_{11}W_\delta)^{-1}V_{23}\right].$$

Finally we can take up (**??**). At this point it is easy to see that if $D_{11}W_\delta$ exists then the right hand side of (**??**) can be reduced (through uniform converge in T) to a new operator: $f_\infty^{22}\colon C^0(X\times\Omega; M(n\times n)) \to C^0(X\times\Omega; M(n\times n))$ which is defined as

$$(A.9)\qquad (f_\infty^{22}M)(x,\omega) = V_{33}+\delta M-(V_{32}+\delta D_{21}W_\delta)(V_{22}+\delta D_{11}W_\delta)^{-1}(V_{23}+\delta D_{12}W_\delta)$$

where the functions on the right hand side are evaluated as follows: $V_{ij}=V_{ij}(x,\tau_\delta(x,\omega);\omega)$, $M=M(\tau_\delta(x,\omega);\omega)$, $D_{ij}W_\delta = D_{ij}W_\delta(\tau(x,\omega);\omega)$. We have:

LEMMA A.3. The operator f_∞^{22} from $C^0(X\times\Omega; M(n\times n))$ into itself is a contraction.

Proof. That f_∞^{22} is well defined from $C^0(X\times\Omega; M(n\times n))$ into itself should be clear by construction. That it is a contraction can be seen by taking any pair M^1, $M^2\in C^0(X\times\Omega; M(n\times r))$ and computing:

$$\|f_\infty^{22}M^1 - f_\infty^{22}M^2\| \le \delta\|M^1-M^2\|.$$

□

Our discussion can therefore be concluded with:

Theorem A.1. Under the hypotheses (A.1), (A.2*) and (A.3) the value function W_δ is of class $C^2(X\times\Omega;\mathbb{R})$ if the discount factor satisfies the following:

$$\delta L^2 < \min\left\{[\beta+\delta\alpha^*(\delta)], [\beta+\delta\alpha^*\delta]^2\right\}.$$

Proof. By appropriately normalizing units we can have $L=\sup_{(x,y)\in D}\|V_{12}\|>1$. Then $L^2>l$. The conclusion then follows from Theorem **??** in the main text, Lemma **??** and the Corollary to Lemma **??**. □

The Algebra of (P_ω): The first order conditions at state T are:

$$V_2(x,\theta_T(x,\omega);\omega)+\delta D_1 W_{T-1}(\theta_T(x,\omega);\omega)=0$$
$$x\in\mathbb{R}^n,\quad \omega\in\mathbb{R}^m,\quad \theta\colon \mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n.$$

Denote with:

$$z=(x,\omega)\in\mathbb{R}^{n+m},\quad F\colon \mathbb{R}^{n+m}\times\mathbb{R}^n\to\mathbb{R}^n$$
$$F(z,\theta(z))=0 \qquad \Leftrightarrow \text{FOC's.}$$

Then:

$$D_z F\left(z, \theta(z)\right) = [V_{21}; V_{23} + \delta D_{12} W_{T-1}] \colon I\!R^{n+m} \to I\!R^n$$
$$D_\theta F\left(z, \theta(z)\right) = [V_{22} + \delta D_{11} W_{T-1}] \colon I\!R^n \to I\!R^n$$
$$D_z \theta(z) = [D_1\theta; D_2\theta] \colon I\!R^{n+m} \to I\!R^n.$$

We write: $D_z F(\bullet, \bullet) + D_\theta F(\bullet, \bullet) D_z \theta(\bullet) = 0$, which is:

$$[V_{21}; V_{23} + \delta D_{12} W_{T-1}] + [V_{22} + \delta D_{11} W_{T-1}][D_1\theta_T; D_2\theta_T] = 0.$$

Now: $[V_{22} + \delta D_{11} W_{T-1}]^{-1}$ exists for the strong concavity assumption. Therefore:

$[D_1\theta_T(z); D_2\theta_T(z)] = -[V_{22} + \delta D_{11} W_{T-1}]^{-1}[V_{21}; V_{23} + \delta D_{12} W_{T-1}]$. Use the notation: $L_{T-1}^{-1} = [V_{22} + \delta D_{11} W_{T-1}]^{-1}$ to get:

$$D_1\theta_T(x, \omega) = -L_{T-1}^{-1}[V_{21}]\, D_2\theta_T(x, \omega) = -L_{T-1}^{-1}[V_{23} + \delta D_{12} W_{T-1}].$$

From totally differentiating: $W_T(x, \omega) = V\left(x; \theta_T(x, \omega); \omega\right) + \delta W_{T-1}\left(\theta_T(x, \omega); \omega\right)$ and applying the "envelope theorem" we get:

$$D_z W_T(z) = [D_1 W_T; D_2 W_T] = [V_1; V_3 + \delta D_2 W_{T-1}].$$

Once again we need to take the derivative of $D_z W_T(z)$ with respect to the all $z = (x, \omega)$. This will give

$$D_{zz} W_T(z) = \begin{bmatrix} D_{11} W_T & D_{12} W_T \\ D_{21} W_T & D_{22} W_T \end{bmatrix}$$
$$= \begin{bmatrix} V_{11} + V_{12} D_1\theta_T & V_{13} + V_{12} D_2\theta_T \\ V_{31} + (V_{32} + \delta D_{21} W_{T-1}) D_1\theta_T & V_{33} + \delta D_{22} W_{T-1} \\ & \quad + (V_{32} + \delta D_{21} W_{T-1}) D_2\theta_T \end{bmatrix}.$$

By substituting in the value of $D_z\theta_T$ found above, we can see that:

$$D_{11} W_T = V_{11} - V_{12} L_{T-1}^{-1} V_{21},$$
$$D_{12} W_T = V_{13} - V_{12} L_{T-1}^{-1}(V_{23} + \delta D_{12} W_{T-1}) = \text{transpose of } D_{21} W_T,$$
$$D_{22} W_T = V_{33} + \delta D_{22} W_{T-1} - (V_{32} + \delta D_{21} W_{T-1}) L_{T-1}^{-1}(V_{23} + \delta D_{12} W_{T-1}).$$

The first one (i.e., $D_{11}W_T$) is obtained by the usual sequence of operator $\{f_T\}_{T=0}^{\infty}$. We know they converge uniformly to f_∞, so that reduces to the well known mapping:

$$(f_\infty \phi)(x,\omega) = [V_{11} - V_{12}L_\phi^{-1}V_{21}](x,\omega).$$

With respect to $D_{12}W_T$ we can compute (set $L_\phi^{-1} = L^{-1}$):

$$\begin{aligned} D_{12}W_T &= V_{13} - V_{12}L^{-1}\left(V_{23} + \delta\left(V_{13} - V_{12}L^{-1}(V_{23} + \delta D_{12}W_{T-2})\right)\right) \\ &= [I - \delta V_{12}L^{-1}][V_{13} - V_{12}L^{-1}V_{23}] \\ &\quad + \delta^2 V_{12}L^{-1}V_{12}L^{-1}[V_{13} - V_{12}L^{-1}(V_{23} + \delta D_{12}W_{T-3})] \\ &= [I - \delta V_{12}L^{-1} + \delta^2 V_{12}L^{-1}V_{12}L^{-1}][V_{13} - V_{12}L^{-1}V_{23}] \\ &\quad - \delta^3 V_{12}L^{-1}V_{12}L^{-1}V_{12}L^{-1}D_{12}W_{T-3}. \end{aligned}$$

Let's simplify notation! $V_{12}L^{-1} = A$ ($n \times n$ matrix); $V_{13} - V_{12}L^{-1}V_{23} = B(n \times n)$. Then we have (by recursive substitution):

$$D_{12}W_T = \sum_{i=0}^{T-1}(-\delta A)^i B \pm (\delta A)^T D_{12}W_0.$$

Therefore if the matrix power $(\delta V_{12}L^{-1})^T \to 0$ as $T \to +\infty$ our sequence will converge. It is easy to see that $\rho_M(\delta V_{12}L^{-1}) = \|\delta V_{12}L^{-1}\| \leq \delta\|V_{12}\|\ \|L^{-1}\| \leq \delta L \cdot [\beta + \delta\alpha^*(\delta)]^{-1}$. Therefore the sequence will converge as long as $\delta L < \beta + \delta\alpha^*(\delta)$ is satisfied. The previous argument provides the proof to **??**. We have taken explicit limits here in order to compute $D_{12}W_\delta$ in terms of V_{ij} and $D_{11}W_\delta$ (see the Corollary to Lemma **??**). To show that f_∞^{12} is a contraction one may simply check that:

$$\|f_\infty^{12}M^1 - f_\infty^{12}M^2\| \leq \delta\|V_{12}L_\phi^{-1}\|\ \|M^1 - M^2\|$$

for any pair M^1, M^2 in $C^0(X \times \Omega; M(n \times m))$.

B Some Examples

Example 1: Let $V(x,y)$ be equal to $-4x^2y+4xy-1/2y^2-R/2\,x^2$, $X=[0,1]$ and $D=X\times X$. It is certainly concave for $R>16$, and $\psi(\mu)$ is equal to $R-16(1+\mu)^{-1}$. Also we have that $V(\bullet,y)$ is R-concave for all $y\in[0,1]$ and $V(x,\bullet)$ is 1-concave for all $x\in[0,1]$. Finally $L=\sup|C_{12}|=4$. This V produces chaotic τ for small δ's. From setting $\psi(\delta\alpha)\ \alpha$ and taking the positive root we get:

$$\alpha^*(\delta)=\left[R\delta-1+\left(1+R^2\delta^2+2R\delta-64\delta\right)^{1/2}\right](2\delta)^{-1}.$$

Now $\left[\beta+\delta\alpha^*(\delta)\right]^2>\delta L^2$ reduces to:

$$64\delta<\left[1+R\delta+\left(1+R^2\delta^2+2R\delta-64\delta\right)^{1/2}\right]^2.$$

The term $(1+R^2\delta^2+2R\delta-64\delta)$ is always nonnegative for $\delta\in[0,1]$ and $R\geq 16$. But then $(1+R\delta)^2-64\delta>0$ would suffice, which is always true as long as $R>16$, i.e., as long as V is strongly concave. One may not assume that if the criterion $\delta L^2\ /\ \beta^2<1$ (see Theorem **??**) had to be applied here we would get the result only for $\delta\in[0,.0625]$! This function V also satisfied the requirements for smoothness with respect to parameters (see Appendix C). **Example 2:**